Math 305: Linear Algebra Homework Packet

Created by: Professors Amanda Harsy and Michael Smith
Lewis University

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Instructions: Please write your solutions **on** these homework pages and show enough of your work so that we can follow your thought process. This makes it easier for us to grade. Follow the instructions for each question. If we can't read your work or answer, you will receive little or no credit!

Some of the homework assignments require you to watch a video or read something beforehand.

For problems 1-3, you must do Gaussian Elimination by hand.

1. Solve the system of equations using Gaussian Elimination by hand. Make sure you continue with the process until your augmented system in RREF form (Row Reduced Echelon Form). Write the solution set. You must show your steps.

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 4 \\ 2x & - & y & + & 3z & = & 2 \\ -x & + & y & - & 2z & = & -1 \end{array}.$$

2. Solve the system of equations using Gaussian Elimination by hand. Make sure you continue with the process until your augmented system in RREF form (Row Reduced Echelon Form). Write the solution set. You must show your steps.

$$\begin{array}{rcrcrcr} x & - & 2y & - & 3z & = & 2 \\ 4x & + & y & - & 2z & = & 8 \\ 5x & - & y & - & 5z & = & 10 \end{array}.$$

3. Solve the system of equations using Gaussian Elimination by hand. Make sure you continue with the process until your augmented system in RREF form (Row Reduced Echelon Form). Write the solution set. You must show your steps.

$$\begin{array}{rcrcrcl} x & - & 2y & - & 3z & = & 2 \\ 4x & + & y & - & 2z & = & 8 \\ 5x & - & y & - & 5z & = & 3 \end{array}.$$

4. Determine whether the following matrices are in echelon form, reduced echelon form, or not in echelon form:

a) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ Circle one: echelon form, reduced echelon form, not in echelon form

b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -7 \end{bmatrix}$ Circle one: echelon form, reduced echelon form, not in echelon form

c) $\begin{bmatrix} -10 & 1 & 1 & 0 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$ Circle one: echelon form, reduced echelon form, not in echelon form

d) $\begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ Circle one: echelon form, reduced echelon form, not in echelon form

5. A system of equations is called **homogeneous** if the right-hand-sides of each equation is 0. So it looks something like this:
$a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4 = 0$
$a_2x_1 + b_2x_2 + c_2x_3 + d_2x_4 = 0$
$a_3x_1 + b_3x_2 + c_3x_3 + d_3x_4 = 0$
$a_4x_1 + b_4x_2 + c_4x_3 + d_4x_4 = 0$

   Eva looks at this and says she knows what will always be a solution to a homogeneous system of equations. What is it and why?

6. Archer says "A system with more unknowns than equations has at least one solution." Eva isn't so sure about this statement. Help her find a counterexample to disprove the statement Archer made.

7. Eva and Archer, the Magnificent Math Kitties are wanting to find the coefficients a, b, and c so that the graph of $f(x) = ax^2 + bx + c$ passes through the points $(1, 2), (-1, 6)$, and $(2, 3)$. Eva says we can set up a system of equations to solve this problem, but Archer says that we can't since we have a quadratic function and not a linear function? Who is right? If Eva is correct, set up and use Gaussian Elimination to solve this augmented matrix. Hint: Eva is usually correct.

8. Given the system below, what value could k and h be so that the system has infinitely many solutions:

$$-x + 8y = h \tag{1}$$
$$-4x + ky = 2 \tag{2}$$

9. Given the system below, what value could k and h be so that the system has no solutions:

$$-x + 8y = h \tag{3}$$
$$-4x + ky = 2 \tag{4}$$

10. For the system below to have 1 solution what restrictions (if any) do we need on k and h?

$$-x + 8y = h \tag{5}$$
$$-4x + ky = 2 \tag{6}$$

11. The Spectacular Math kitties, Eva and Archer are trying to solve the following matrix equation for X:

$$\begin{bmatrix} -5 & -5 \\ -2 & 2 \end{bmatrix} \cdot X + \begin{bmatrix} -2 & 7 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} -7 & -6 \\ 5 & 1 \end{bmatrix} \cdot X$$

Archer first rearranges the equation so the X is on one side:

$$\begin{bmatrix} -2 & 7 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} -7 & -6 \\ 5 & 1 \end{bmatrix} \cdot X - \begin{bmatrix} -5 & -5 \\ -2 & 2 \end{bmatrix} \cdot X$$

Eva says we can factor out the X to get

$$\begin{bmatrix} -2 & 7 \\ 8 & 5 \end{bmatrix} = (\begin{bmatrix} -7 & -6 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} -5 & -5 \\ -2 & 2 \end{bmatrix}) \cdot X$$

This gives us

$$\begin{bmatrix} -2 & 7 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 7 & -1 \end{bmatrix} \cdot X$$

Now Archer isn't sure how to solve for X. When we solve $3 = 2x$ for x we would just divide both sides of the equation by 2, but he knows we don't have matrix division (and haven't learned inverses... yet). Eva suggests writing X as a matrix with variables which can help u set up an equation to solve for the entries of X.

a) Eva says X must be a 2 by 2 matrix. Why is this so?

b) After letting $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, set up and solve a system of equations to determine X:

$$\begin{bmatrix} -2 & 7 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 7 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Using Octave to Row Reduce Matrices:

Use the octave/MatLab operation rref to row reduce matrices. Suppose you want to row reduce (Use Gaussian Elimination) the following augmented matrix: $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right)$

You can type:
B=[1 -2 1 0; 0 2 -8 8; 5 0 -5 10] or B=[1,-2, 1, 0; 0, 2, -8, 8; 5, 0, -5, 10]
Then type: rref(B)
which will spit out this matrix:

$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}$$

Which means the solution to the system is $(1, 0, -1)$.

For the following linear systems, use Octave or Matlab to row reduce these systems, write the row reduced echelon form matrix given by octave/Matlab and write the solution set, if it exists.

12. $\begin{array}{rcrcrcr} x & - & y & - & z & = & 4 \\ 2x & - & y & + & 3z & = & 2 \\ -x & + & y & - & 2z & = & -1 \end{array}.$

13. $\begin{array}{rcrcrcr} x & - & 2y & - & 3z & = & 2 \\ 4x & + & y & - & 2z & = & 8 \\ 5x & - & y & - & 5z & = & 10 \end{array}.$

14. $\begin{array}{rcrcrcr} x & - & 2y & - & 3z & = & 2 \\ 4x & + & y & - & 2z & = & 8 \\ 5x & - & y & - & 5z & = & 3 \end{array}.$

15. In this problem, we are going to use homogeneous linear equations to balance the chemical equation $NH_3 + O_2 \rightarrow NO + HO_2$. To balance the equation, you must determine the number of moles x_k of each chemical species to use:

$$x_1(NH_3) + x_2(O_2) \rightarrow (x_3)NO + (x_4)HO_2$$

a) Set up an augmented matrix equation of the form $Ax = 0$ with $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ which can be used to solve this equation. **Note: If you need help, I have created a video called Balancing Equations under Homework.**

b) Suppose $x_4 = 20$ moles. Determine x_1, x_2, and x_3.

c) What value should be pick for x_4 to balance the equation without fractions?

16. Determine the additive inverse of the vector in $\mathcal{P}_3$, $-3-2x+x^2$.

17. Determine the additive inverse of the vector in $\mathcal{M}_{2\times 2}$, $\begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix}$.

18. For the next few problems, decide if the vector lies in the span of the set. If it does, find the linear combination that makes the vector. If it does not, show that no linear combination exists.

    (a) Vector= $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^3$.

(b) vector $= x - x^3$, span $\left\{x^2, 2x + x^2, x + x^3\right\}$, in $\mathcal{P}_3$

(c) vector $= \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$, span $\left\{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}\right\}$, in $\mathcal{M}_{2\times 2}$

Using Octave to Determine Spanning Sets:

During you last HW we used octave/MatLab operation rref to row reduce matrices that only have numerical values. It turns out that Octave/MatLab can also row reduce matrices with variables, **but you must tell Octave/Matlab to treat them like symbolic variables.** To do this, you can type "syms x" (no quotes) to make x a symbolic variable and or for multiple variables you can type "syms x y z" (without quote marks) you could also define the variable cat, by writing "cat=sym('cat')". For example

Suppose I want to show that $\mathbb{R}^3$ is spanned by $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$. Remember this means I want to pick an arbitrary vector in $\mathbb{R}^3$, say $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and show I can write this as a linear combination of the set above. To do this, I set up the vector equation: $\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and solve for a, b, & c. So we want to solve the system:

$$\begin{array}{ccccccc} \alpha & + & 0 & + & 0 & = & x \\ \alpha & + & 2\beta & + & 0 & = & y \\ \alpha & + & 2\beta & + & 3\gamma & = & z \end{array}.$$

To solve this in octave, first define x, y,& z as variables by typing: syms x y z
Then define the augmented equations we want to solve for by typing:
B=[1 0 0 x; 1 2 0 y; 1 2 3 z] or B=[1,0, 0, x; 1, 2, 0, y; 1, 2, 3, z]
Then type: rref(B) or rref([1 0 0 x; 1 2 0 y; 1 2 3 z])
which will spit out this matrix:

$$\begin{array}{cccc} 1 & 0 & 0 & x \\ 0 & 1 & 0 & -\frac{x}{2} + \frac{y}{2} \\ 0 & 0 & 1 & -\frac{y}{3} + \frac{z}{3} \end{array}$$

Thus this set is a spanning set for $\mathbb{R}^3$ since $\alpha = x, \beta = -\frac{x}{2} + \frac{y}{2}, \gamma = -\frac{y}{3} + \frac{z}{3}$.
Note: If you want Octave to spit out rational numbers type rats(rref(B)) or rats(rref([1 0 0 x; 1 2 0 y; 1 2 3 z]))

So for example, if you give me the vector $\begin{bmatrix} 7 \\ 3 \\ 9 \end{bmatrix}$, I define $\alpha = 7, \beta = -2, \gamma = 2$.

19. For the following problems, use Octave/Matlab to determine whether the following sets span $\mathbb{R}^3$. Remember you need to pick an arbitrary element in $\mathbb{R}^3$ and see if you can write it as a linear combination of the set of vectors.
    For each problem. a) Type the row reduced echelon matrix found by Octave.
    b) Tell me if this set spans $\mathbb{R}^3$ and if so, solve for $\alpha, \beta, ...$
    c) Otherwise, explain how can you tell when there is no possible linear combination?
    d) If the set spans $\mathbb{R}^3$, what could you define α, β, γ, etc. to construct the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

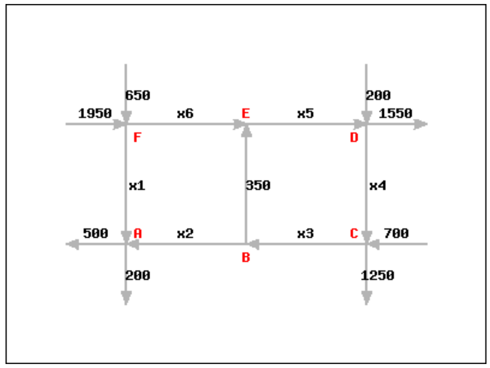
    (a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$

    (b) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} \right\}$

20. Consider the network diagram:



This can be modeled using a linear system of the form $Ax = 0$ with $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{bmatrix}$. At each data point, create an equation that maintains equilibrium. That is the input equals the output.

a) Set up an augmented matrix of the form $Ax = 0$ which can be used to solve this equation. **Note: If you need help, I have created a little video called "'Network Flow" with an example similar to this the Homework Tab of Blackboard.**

b) Suppose we know $x_6 = 2100$, determine what x_1, x_2, x_3, x_4, and x_5 are.

21. **Scenario Three: Hide-and-Seek**

Recall, Eva and Archer can travel the world using their hover board and magic carpet which both have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

In Ice 2, we found that Eva and Archer could use these items to travel to Mako's cabin which was located 107 miles East and 64 miles North of their home. **Mako wants to move to a cabin in a different location. Eva and Archer are not sure whether Mako is just trying to test their wits at finding him or if he actually wants to hide somewhere that they can't visit him.**

Question: Are there some locations that Mako can hide and Eva and Archer cannot reach him with these two modes of transportation?Include a convincing argument supporting your answer.

22. Suppose Eva and Archer are now in a three-dimensional world for the carpet ride problem, and they have three models of transportation: $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$ Is there anywhere in this 3D world that Mako could hide from Eva and Archer? If so, where? If not, why not?

23. Eva and Archer are trying to determine if the set $V = \{a_0 + a_1 x + a_2 x^2 \mid a_0 + 2a_1 + a_2 = 4\}$ a subspace of $\mathcal{P}_2$. Eva says it is pretty clear that it is missing an important aspect for being a subspace. What is this set missing?

24. Bonus: Archer wants to change the set only a little to make it a subspace. What could he do?

25. Which of these subsets are subspaces of $\mathcal{M}_{2\times 2}$? For each one that is a subspace, write the set as a span. For each that is not, show the condition that fails.

(a) $\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \; a+b=0\right\}$

(b) $\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \; a+b=5\right\}$

(c) $\left\{\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \middle| \; a+b=0, c\in\mathbb{R}\right\}$

26. A dietitian is planning a meal that supplies certain quantities of vitamin C, calcium, and magnesium. Three foods will be used, their servings measured in milligrams. The nutrients supplied by these foods and the dietary requirements are given in the table below.

Nutrient	Food 1	Food 2	Food 3	Total Required (mg)
Vitamin C	30	45	15	1785
Calcium	20	45	20	1642.5
Magnesium	30	60	35	2472.5

a) Set up and solve an augmented matrix of the form $Ax = b$ which can be used to answer the questions below.

b) What servings (mg) of Food 1 is necessary to meet the dietary requirements?

c) What servings (mg) of Food 2 is necessary to meet the dietary requirements?

d) What servings (mg) of Food 3 is necessary to meet the dietary requirements?

Bonus Questions: Galois Field 2

Listen to the youtube video on my channel called Galois Field 2 and then answer the following questions. The YouTube link can be found here: https://youtu.be/rfC8ctixQKw. (you may need to copy and paste this into your browswer)

27. Calculate each of the following expressions over $GF(2)$:
a) $1 + 1 + 1 + 0 =$

b) $1 \cdot 1 + 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 =$

c)$(1 + 1 + 1) \cdot (1 + 1 + 1 + 1) =$

d) $1 - 1 - 1 - 1 - 1 - 1 - 1 - 0 =$

e) $(1 - 1) \cdot (0 - 1) =$

28. List all 8 vectors in $GF(2)^3$

29. Show that the set of all possible linear combinations of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ over the field $GF(2)$ is $GF(2)^2$. That is, show Span($\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$)=$GF(2)^2$

30. How many vectors are in the Span($\begin{bmatrix} 1 \\ 1 \end{bmatrix}$) over $GF(2)$ (That is, our scalar multiples are either 0 or 1)?

31. Why might the field $GF(2)$ be interesting to you?

32. Eva and Archer are asked to determine if the following sets form a basis for $\mathbb{R}^3$. Eva says that they only really need to check one of the sets and can immediately tell that the other two sets can't be a basis for $\mathbb{R}^3$. Pick out the two sets that can be eliminated and help Eva explain why they can be eliminated. **Note, you do not have to check if any of the sets are bases for** $\mathbb{R}^3$.

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

33. Determine which of the following sets is a basis for $\mathcal{P}_2$. Remember to check that the set is a linear independent set AND that it spans $\mathcal{P}_2$.

   (a) $\mathcal{A} = \{x^2 - 1, 1 + x, 1 - x\}$

Next Question is on the back

(b) $\mathcal{B} = \{1, 1 - x, 1 + x, 1 + x^2\}$

34. Determine whether the following set is a basis for $\mathcal{M}_{2\times 2}$. Remember to check that the set is a linear independent set AND that it spans $\mathcal{M}_{2\times 2}$.

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

35. Consider the subspace $V = \{\begin{pmatrix} a & b & c \\ 0 & b-c & 2a \end{pmatrix} | a, b, c \in \mathbb{R}\} \subseteq \mathcal{M}_{2\times 3}$

a) Write the this subspace as a span.

b) Show this spanning set is a basis for V by showing that it is linear independent.

c) What is the dimension of V?

36. **Bonus Question:** Suppose $\{v_1, v_2, v_3, v_4\}$ are linearly independent. Determine which of the following sets is a basis for span$\{v_1, v_2, v_3, v_4\}$. Justify your answer.

(a) $\{v_1, v_2\}$

(b) $\{v_1, v_2, v_3, v_4, v_1 - 2v_3\}$

(c) $\{v_1 + v_3, v_2 + v_4, v_3, v_4\}$

37. The Amazing Math Kitties, Archer and Eva are doing an experiment which produces the following data: $(1, 1.8), (2, 2.7), (3, 3.4), (4, 3.8)$. [Note these data points are in the form (x, y)]. The kitties would like to fit this data to the parabolic function $y = \alpha x^2 + \beta x$. In other words they want to solve for α and β which will give them a best-fit curve for this data.
a) Help Eva and Archer by setting up a matrix equation $A\mathbf{x} = \mathbf{b}$ for this system and use octave to show that this system is inconsistent.

b) Archer wonders what they should do since this system has not solution. Eva says, "Purrrhaps we can find an approximate solution..." Archer says he remembers in class we learned how to find a least-squares solution to inconsistent systems. Help the kitties find such a least-squares solution for this system and **write out the best-fit parabola you get**.

38. A healthy child's systolic blood pressure p (in millimeters of mercury) and weight w in pounds are approximately related by the equation $\alpha_0 + \alpha_1 \ln(w) = p$.

a) Use the experimental data in the table below to find the best-fit function which relates the data. Note Eva calculated $\ln(w)$ for you!

Weight (w **in lbs**)	**ln(w)**	**Pressure** (p **in ml**)
44	3.78	91
61	4.11	98
81	4.39	103
113	4.73	110
131	4.88	112

b) Use your function to estimate the systolic blood pressure of a healthy child weighing 100 lbs.

39. Suppose T is a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}$.

a) Find a vector $\mathbf{x}$ (if it exists) whose image under T is $\mathbf{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$

b) Is $\mathbf{x}$ unique?

40. Define $f : \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ by $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-2 \\ b+1 \\ c \\ d \end{pmatrix}$.

(a) What is the image of $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ under f?

(b) Prove/Justify whether or not f is a linear transformation using the definition of linear transformation.

(c) If f is a linear transformation. Find a matrix representation of it.

41. Define $\mathcal{G} : \mathcal{P}_2 \to \mathcal{M}_{2\times 2}$ by $\mathcal{G}(ax^2 + bx + c) = \begin{pmatrix} a & -b \\ 0 & c+3a \end{pmatrix}$.

(a) What is the image of $3x^2 + 2x - 1$ under $\mathcal{G}$?

(b) Prove/Justify whether or not $\mathcal{G}$ is a linear transformation using the definition of linear transformation.

(c) If $\mathcal{G}$ is a linear transformation. Find a matrix representation of it.

42. Define $g : \mathcal{M}_{2\times 2} \to \mathbb{R}^2$ by $g\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b \\ a-c \end{pmatrix}$.

(a) What is the image of $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ under g?

(b) Prove/Justify whether or not g is a linear transformation using the definition of linear transformation.

(c) If g is a linear transformation. Find a matrix representation of it.

43. Define $g : V \to \mathbb{R}^3$, where $V = \mathcal{P}_1$ by $g(ax+b) = \begin{pmatrix} 0 \\ b-a \\ a+b \end{pmatrix}$.

(a) Find the nullspace of g.

(b) Find the nullity of g.

(c) Find the range space of g. (Write it as a span)

(d) What is the rank of g?

(e) Use a nullity argument to determine whether g injective.

(f) Use a rank argument to determine whether g surjective.

(g) Find a matrix representation of $g : V \to \mathbb{R}^3$, where $V = \mathcal{P}_1$ by $g(ax + b) = \begin{pmatrix} 0 \\ b - a \\ a + b \end{pmatrix}$.

(h) Find a basis for the Column Space of this matrix.

44. Define $f : \mathcal{P}_2 \to \mathbb{R}^2$ by $f(ax^2 + bx + c) = \begin{pmatrix} a+b \\ a-c \end{pmatrix}$.

(a) Find the nullspace of f.

(b) Find the nullity of f.

(c) Find the range space of f. (Write it as a span)

(d) What is the rank of f?

(e) Use a nullity argument to determine whether f injective.

(f) Use a rank argument to determine whether f surjective.

(g) Find a matrix representation of $f : \mathcal{P}_2 \to \mathbb{R}^2$ by $f(ax^2 + bx + c) = \begin{pmatrix} a+b \\ a-c \end{pmatrix}$.

(h) Find a basis for the Column Space of this matrix.

45. Define $T : \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ 3c \\ -d \end{pmatrix}$.

(a) Find the nullspace of T.

(b) Find the nullity of T.

(c) Find the range space of T. (Write it as a span)

(d) What is the rank of T?

(e) Use a nullity argument to determine whether T injective.

(f) Use a rank argument to determine whether T surjective.

(g) Find a matrix representation of $f : \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ by $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ 3c \\ -d \end{pmatrix}$.

(h) Find a basis for the Column Space of this matrix.

46. For each of the following matrices, find the Column space of M, rank$M$, nullity(M), the dimension of M (example a 2 by 3 matrix), the number of columns without leading entries, and the number of leading entries in the echelon form.

(a) $A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -1 \\ 2 & 3 & 2 & -2 \\ 5 & 6 & -1 & -5 \end{pmatrix}$ The row reduced echelon form of A is $\begin{pmatrix} 1 & 0 & -5 & -1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

The nullity of $A =$

The rank of A=

A basis for the column space of A is

The number of columns without leading entries is

The number of leading entries (pivots) in the echelon form is

Is the linear transformation represented by A injective, surjective, bijective?

Please continue to part b of this question on next page!

(b) $B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. The row reduced echelon form of B is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

The nullity of $B =$

The rank of B=

A basis for the column space of B is

The number of columns without leading entries is

The number of leading entries (pivots) in the echelon form is

Is the linear transformation represented by B injective, surjective, bijective?

(c) $C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \\ 3 & 1 & 4 \\ -1 & 0 & 1 \end{pmatrix}$ The row reduced echelon form of C is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The nullity of C=

The rank of C=

A basis for the column space of C is

The number of columns without leading entries is

The number of leading entries (pivots) in the echelon form is

Is the linear transformation represented by C injective, surjective, bijective?

47. Compute $\begin{vmatrix} 1 & 5 & 3 \\ 2 & -3 & 0 \\ -1 & 1 & 0 \end{vmatrix}$ by hand. You can check your answer using Octave.

48. Compute the determinant of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ -1 & 1 & -2 \end{bmatrix}$ by hand using two different expansions. Show your expansion/work.

49. Find the eigenvalues and a basis for for the eigenspace of the matrix associated with each eigenvalue for $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$.

50. Find the eigenvalues and a basis for for the eigenspace of the matrix associated with each eigenvalue for $B = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Bonus Questions: Cramer's Rule

Good for Physics Majors

Cramer's Rule is a nice application of determinants which is often used in engineering and physics to study how solutions to $Ax = b$ is affected by changing $\mathbf{b}$.
Cramer's Rule: If A is an invertible, $n \times n$ matrix, for any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution of the system $Ax = b$ is given by: $x_i = \frac{|A_i(b)|}{|A|}$. Note $|A_i(b)|$ is the determinant of A with A's i^{th} column replaced with b.
Watch the short 3 minute youtube video https://www.youtube.com/watch?v=PO4hpSyxH9g demonstrating Cramer's Rule. (If you want to see a 3 by 3 system example go here: https://www.youtube.com/watch?v=UMrg5Xt7gGA):

51. Use Cramer's rule to compute the solution to the system below. Find $D = |A|$, $D_x = |A_x(b)|$, $D_y = |A_y(b)|$
$4x + y = 6$
$3x + 2y = 7$

52. Use Cramer's rule to compute the solution to the system below. Find $D = |A|$, D_x, D_y, and D_z. Remember you can use octave to find determinants.
$x + y = 3$
$-3x + 2z = 0$
$y - 2z = 2$

53. Eva and Archer are trying to compute A^5 for some diagonalizable 2 by 2 matrix A. The only things that the kittes know about A are that A has an eigenspace associated with eigenvalue 1 that is $\mathcal{E}_1$ =span$\{\begin{bmatrix} 2 \\ 7 \end{bmatrix}\}$ and an eigenspace associated with eigenvalue -3 that is $\mathcal{E}_{-3}$ =span$\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\}$. Can Eva and Archer determine what A^5 is? If so, fnd A^5. If not, explain why it is impossible.

54. Eva and Archer are trying to diagonalize a 4 by 4 matrix A. Eva has found 3 unique eigenvalues for A. Archer says this means A is not digaonalizable since we do not have 4 unique eigenvalues. Is he correct? If not, help Eva correct his reasoning.

55. Diagonalize $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ (by finding P and D) or explain why B isn't diagonalizable.

56. It has been claimed that the best predictor of today's weather is today's weather. Suppose that in the town of Octapa, if it rained yesterday, then there is a 60% chance of rain today, and if it did not rain yesterday there is an 85% chance of no rain today.

a) Find the transition matrix describing the rain probabilities.

b) If it rained Monday, what is the probability it will rain on Wednesday?

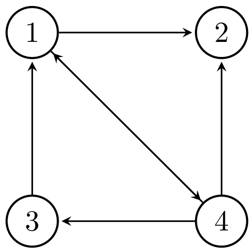
c) If it did not rain Friday, what is the probability of rain on Monday?

d) Using the transition matrix from part a, find the steady-state vector. Use this to determine the probability that it will be raining at the end of time.

57. Diagonalize $C = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ (by finding P and D) or explain why C isn't diagonalizable.

58. Below is a directed graph representing four web pages representing links to other pages and you have equal probability to any page in the network. Assume there is a 40% chance of clicking to a random page.

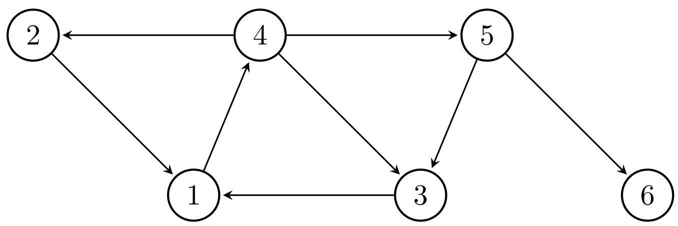
a) Create the Transition Matrix for this Markov process.



b)Find the unique steady state vector and use it to determine a ranking for these pages.

59. Below is a directed graph representing six web pages representing links to other pages. For Example Page 5 links to 6 and 3. 6 is a dangling node and you have equal probability to any page in the network. Assume there is a 30% chance of clicking to a random page.

a) Create the Transition Matrix for this Markov process. **Hint: Try to keep entries exact (fractions, not decimals).**



b)Find the unique steady state vector and use it to determine a ranking for these pages. **Hint: Try to keep entries exact (fractions, not decimals). Use *rats* in octave.**

60. Find the QR factorization by hand for $A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$

61. True or False: $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$ is an orthogonal matrix. If False, explain.

62. True or False: $Q = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is an orthogonal matrix. If False, explain.

63. What is the difference between rank and numerical rank?

64. Suppose you want to find the singular value decomposition for a **4 by 2** matrix A. Dr. Harsy has computed the following eigenvalues and associated eigenvectors for A^TA: $\lambda_1 = 3$ with associated eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 16$ with associated eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Determine what Σ and V^T are for the singular value decomposition, if it is possible or explain why we can't find this decomposition.

65. Please complete the Lewis University evaluation for my Applied Linear Algebra class. You should receive an email from noreply@tk20.com or caleighaoconnell@lewisu.edu with a link to do the evaluations. If 90% of the class has completes the evaluation, I will drop one homework assignment. Please give thoughtful, constructive feedback which will help me improve the course. For example, saying "You suck" or "You are great" doesn't provide much feedback for me. Saying "You suck because..." or "You are great because..." Also, remember for everything you like about the course, there is at least one other person who dislikes it, so please let me know what you would like to be kept the same about the course.

    Check one:

    ☐ I completed the evaluation.
    ☐ I have not completed the evaluation.

66. Create a Meme about this course. It can be something about the topic we covered this semester, but it should relate in some way to this course. On the last day of school, we will share all of the memes and vote for the best one. [Note: You can use a meme that already exists if it relates to this course, but it is more fun to create your own.] I will create an assignment in blackboard in which you can upload your Meme or you can print it and attach to this paper. You can find pictures of Eva and Archer if you would like to use them in your meme here: `http://bit.ly/EvaArcherPics` if you want to use them. We will have a contest for whose meme is the best meme!

67. Create a "good" Archer answer relating to something from what we learned in class this semester. So this should be an incorrect answer (but not a *trivial* incorrect answer) that demonstrates a subtle misconception about a concept or topic in this course. Then write what Eva should say in order to help correct his mistake and explain what the misconception is.