MATH 25000: Calculus III Lecture Notes

Created by Dr. Amanda Harsy

November 29, 2023

Contents

1 Syllabus and Schedule

Thanks for taking Calculus III with me! It is one of my favorite classes to teach and I think it is a great way to end your Calculus Sequence. Now, you may be asking yourself (or have asked yourself), "What is Calculus, and why do I have to take this class?" Calculus is, in my opinion, ultimately is the study of *change*. In particular, calculus gives us the tools to be able to understand how changing one or more linked variables reflects change in other variables [1]. In other words, Calculus is the study and modeling of dynamical systems[2]. In Calculus I, we learned about the derivative of a function and some of its applications. Recall, a derivative is a measure of *sensitivity* of change in one variable to change in the other -the instantaneous rate of change. When we learn about integration, we are measuring *accumulation* or the limit of a summation of smaller parts[2]. In Calculus II, we built upon this idea that we can use integrals to calculate and model complex situations by accumulating the sums of simpler parts. We also learned techniques used in calculating and approximating these integrals and discuss ways of modeling functions and infinite systems.

Calculus III should really be renamed, *The Greatest Hits of Calculus*. We revisit all of the amazing theory we learned in Calculus I and II, but now we just generalize it to the multivariate setting. We also generalize it to Vector Fields at the end of the course. At times during this course, the topics may seem disjointed. For example, we start the semester with parametric equations and an introduction to vectors. Differentiation and integration is still there, but isn't the main event during this time. We then get into the greatest hits part of Calc 3 and revisit differentiation and integration. At this point in the course, you may think, wait, but what about the vectors? Don't worry. Our last month will be combining the multivariate calculus with vector calculus and this culminates in several important theorems which tie all of Calculus III topics together into several beautiful and useful packages!

I hope you will enjoy this semester and learn a lot! Please make use of my office hours and plan to work hard in this class. My classes have a high work load (as all math classes usually do!), so make sure you **stay on top of your assignments and get help early**. Remember you can also email me questions if you can't make my office hours or make an appointment outside of office hours for help. When I am at Lewis, I usually keep the door open and feel free to pop in at any time. If I have something especially pressing, I may ask you to come back at a different time, but in general, I am usually available. The HW Assignments, and Practice Problems for Exams are at the end of this course packet. I have worked hard to create this course packet for you, but it is still a work in progress. Please be understanding of the typos I have not caught, and politely bring them to my attention so I can fix them for the next time I teach this course. I look forward to meeting you and guiding you through the magnificent course that is Calculus III.

Cheers,
Dr. H

[1] J. Epstein. 2013 The Calculus Concept Inventory -Measurement of the Effect of Teaching Methodology in Mathematics. *Notices of the AMS* **60** (8), 1018-1026

[2] Schumacher, etc. 2015 *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences* **18**

Acknowledgments: No math teacher is who she is without a little help. I would like to thank my own undergraduate professors from Taylor University: Dr. Ken Constantine, Dr. Matt DeLong, and Dr. Jeremy Case for their wonderful example and ideas for structuring excellent learning environments. I also want to thank the members from both the MathVote Projects for sharing some of their clicker questions. *And finally, I would like to thank you and all the other students for making this job worthwhile and for all the suggestions and encouragement you have given me over the years to improve.*

2 Syllabus Crib Notes

The full syllabus is posted in Blackboard. Here are some highlights from the syllabus:

2.1 Office Hours

Please come to my office hours! Helping you with the material is the best part of my job! I have set weekly office hours which I hold and I encourage you to instead make appointments for me to meet with you at a time that works for both of us! My office is in **Memorial Hall SU 124**. Remember if none of these times work, send me an email and we can schedule another time to meet. I can also answer questions through email!

Mondays: 2:00-3:00
Wednesdays: 2:00-3:00
Fridays: 2:00-3:00
Or By Appointment!

Note: Sometimes I have meetings or class that goes right up to my office hours, so if I am not there, please wait a few minutes. Also sometimes I have unexpected meetings that get scheduled during my office hours. If this happens, I will do my best to let you know as soon as possible and I usually hold replacement office hours.

Help: Don't wait to get help. Visit me during my office hours, use the discussion forum in Blackboard, go to the Math Study Tables, find a study partner, get a tutor!

2.2 Grades

Category	Percentage
Quizzes	10
Homework	15
Online HW	10
Mastery Exams	65

Exam #	Date
1	10/4
2	10/25
3	11/15
4	12/6

Final Exam: We will not have a formal final exam. Instead Finals week will be a final Testing Week (see description above under Master-Based Testing).

Homework: Almost every week, I will collect a homework assignment. I will post these homework assignments on Blackboard. You may work with others on the homework, but it must be your own work. If I catch you copying homework, you will get a 0. Please see the academic honesty section in the posted syllabus in Bb.

2.2.1 Exams

Grading for Growth Assessment (previously called-Based Testing: This course will use a testing method called, "Mastery-Based Testing." There will be four (4) paper-and-pencil, in-class

Mastery Exams given periodically throughout the semester. In mastery-based testing, students receive credit only when they display "mastery", but they receive multiple attempts to do so. The primary source of extra attempts comes from the fact that test questions appear on every subsequent test. In this Calculus III course, Test 1 will have 5 questions. Test 2 will have 10 questions -a remixed version of the five from Test 1 and five new questions. Test 3 will have 15 questions- a remixed version of the ten from Test 2 and five new questions. Test 4 will have 18 questions- a remixed version of the fifteen from Test 3 and three new questions. We will also have four testing weeks. During these weeks, students can sign up to retest concepts during (extended) office hours or during Study Tables hours. Students are allowed to test any concept, but cannot retest that concept the rest of the week. So for example, a student can test concepts 2,3 and 5 on Monday and concept 6 on Tuesday, but would not be able to test concept 5 again. These testing weeks are tentatively scheduled and noted in our course schedule. Our final testing week is during Finals Week.

Grading of Mastery-Based Tests: The objectives of this course can be broken down into 18 main concepts/problems. For each sort of problem on the exam, I identify three levels of performance: master level, journeyman level, and apprentice level. I will record how well the student does on each problem (an M for master level, a J for journeyman level, a 0 for apprentice level) on each exam. After the Final testing week, I will make a record of the highest level of performance the student has made on each sort of problem or project and use this record to determine the student's total exam grade. Each of the first 9 concepts/questions students master will count 8% points towards their exam grade (with the first one being worth 9%). After that, each concept/question will be worth 3% towards your exam grade. So for example, if you master 11 of the 18 concepts your grade will be a 79%.

This particular way of arriving at the course grade is unusual. It has some advantages. Each of you will get several chances to display mastery of almost all the problems. Once you have displayed mastery of a problem there is no need to do problems like it on later exams. So it can certainly happen that if you do well on the midterms you might only have to do one or two problems on the Final. (A few students may not even have to take the final.) On the other hand, because earlier weak performances are not averaged in, students who come into the Final on shaky ground can still manage to get a respectable grade for the course. This method of grading also stresses working out the problems in a completely correct way, since accumulating a lot of journeyman level performances only results in a journeyman level performance. So it pays to do one problem carefully and correctly as opposed to trying to get four problems partially correctly. Finally, this method of grading allows you to see easily which parts of the course you are doing well with, and which parts deserve more attention. The primary disadvantage of this grading scheme is that it is complicated. At any time, if you are uncertain about how you are doing in the class I would be more than glad to clarify the matter.

2.3 Expectations

This is a college level Math class and is much different than one taught in high school. We cover a lot of (very different) material in a very limited class time. You cannot expect to be able to pass this class if you do not spend several hours every day reading the sections and working problems outside of class. Paying attention and taking notes only during class time will not be enough. After the problems are worked, find a common thread, idea, or technique.

Calculator Policy: You will find it useful to have a graphing calculator. I recommend a TI-Nspire CX CAS or a TI-89, but do not buy a new calculator if you already have a close equivalent. The graphing calculators will be allowed on *almost every* test and quiz. No other forms of technology can be used on in-class, closed-books assessments (quizzes, tests, final). The Math Study Tables will have a few TI-Nspires available during the Study Table hours.

Academic Integrity: Scholastic integrity lies at the heart of Lewis University. Plagiarism, collusion and other forms of cheating or scholastic dishonesty are incompatible with the principles of the University. This includes using "tutoring"' sites for homework, quizzes, and exams. Students engaging in such activities are subject to loss of credit and expulsion from the University. Cases involving academic dishonesty are initially considered and determined at the instructor level. If the student is not satisfied with the instructors explanation, the student may appeal at the department/program level. Appeal of the department /program decision must be made to the Dean of the college/school. The Dean reviews the appeal and makes the final decision in all cases except those in which suspension or expulsion is recommended, and in these cases the Provost makes the final decision.

Make-Ups: There will be no make-ups for any assignments. If you are late or miss class, your assignment will not be accepted and there will be no make-up offered, except in extenuating and unpredictable circumstances. If you will miss class for a justifiable & unavoidable reason, you can contact me **before** you miss class & it is possible you can have a make-up. If you do not contact me & explain your absence, you will not be allowed a make-up.

Dr. Harsy's web page: For information on undergraduate research opportunities, about the Lewis Math Major, or about the process to get a Dr. Harsy letter of recommendation, please visit my website: http://www.cs.lewisu.edu/~harsyram.

Blackboard: Check the Blackboard website regularly (at least twice a week) during the semester for important information, announcements, and resources. It is also where you will find the course discussion board. Also, check your Lewis email account every day. I will use email as my primary method of communication outside of office hours.

The full syllabus and schedule is subject to change and the most updated versions are posted in the Blackboard.

Dr. Harsy's Calculus III Schedule Fall 2023

The table below outlines the tentative topics to be covered each day and quiz and homework due dates. You also have Online HW due most Tuesdays and Thursdays. This schedule is subject to change.

Monday	Tuesday	Wednesday	Friday
8/28 Intro, Parametric Eqns	8/29 Parametric Eqns cont.	8/30 3-Space	9/1 Vectors **HW 1 Due**
9/4 **Labor Day, No Class**	9/5 Dot Product	9/6 Cross Product	9/8 Cross Product cont. **HW 2 Due**
9/11 Vector Fns	9/12 Lines and Planes	9/13 Curvature	9/15 Curvature cont. **HW 3 Due**
9/18 Surfaces	9/19 Level Curves	9/20 Polar Coordinates	9/22 Polar graphs **HW 4 Due**
9/25 Calc with polar eqns	9/26 Limits	9/27 Limits cont.	9/29 **Sample Mastery Quiz & Quiz 1** **HW 5 Due**
10/2 Partial Derivatives	10/3 Partial Derivatives cont. with Play Doh	10/4 **Exam 1**	10/6 **Fall Break** **No class**
10/9 Directional Derivatives **(testing week)**	10/10 Linear Approximations **(testing week)**	10/11 Chain Rule **(testing week)**	10/13 Continuity Optimization **HW 6 Due** **(testing week)**
10/16 Optimization	10/17 Lagrange Multipliers	10/18 Lagrange Multipliers cont. **Quiz 2**	10/20 Double Integrals Over Rectangular Regions **HW 7 Due**
10/23 Double Integrals Over General Regions	10/24 Double Integrals Over Polar Regions	10/25 **Exam 2**	10/27 Surface Area Optional Video: Density/Mass
10/30 Triple Integrals	10/31 Triple Integrals Cont.	11/1 Cylindrical Coordinates	11/3 Spherical coordinates **HW 8 due**
11/6 Change of Variables **(testing week)**	11/7 Change of Variables cont. **(testing week)**	11/8 Vector Fields **(testing week)**	11/10 Line Integrals **HW 9 due** **(testing week)**
11/13 Fund Thm of Line Integrals	11/14 Curl and Divergence	11/15 **Exam 3**	11/17 Green's Theorem **HW 10 due**
11/20 Parametric Surfaces	11/21 Surface Integrals **Quiz 3**	11/22 **Thanksgiving Break** **No Class**	11/23 **Thanksgiving Break** **No Class**
11/27 Surface Integrals **(testing week)**	11/28 Stokes Theorem **(testing week)**	11/29 Divergence Theorem **(testing week)**	12/1 Divergence Theorem Cont. **(testing week)**
12/4 Review of Vector Calculus **HW 11 due**	12/5 **Flex** **Quiz 4**	12/6 **Exam 4**	12/8 **Review/Evaluations** **HW 12 due** **Bonus Quiz**
12/11 Finals Week **(testing week)**	12/12 Finals Week **(testing week)**	12/13 Finals Week **(testing week)**	12/14 Finals Week **(testing week)**

No formal final exam. Finals week will be a testing week in which you can arrange times to come in to test concepts.

Assessment and Mapping of Student Learning Objectives:

Baccalaureate Characteristics:

BC 1. The baccalaureate graduate of Lewis University will read, write, speak, calculate, and use technology at a demonstrated level of proficiency.

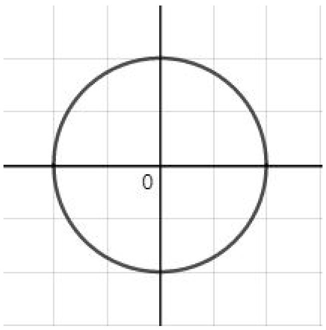
Measurable Student Learning Outcome:

Advocate for a cause or idea, presenting facts and arguments, in an organized and accurate manner using some form of technology. Include qualitative and quantitative reasoning.

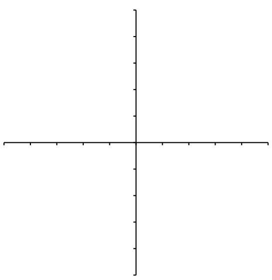
Course Student Learning Outcomes	Baccalaureate Characteristics	Demonstrated By
1. Perform standard operations on vectors in two-dimensional space and three-dimensional space.	1- Reinforced	Homework, Exams, or Quizzes
2. Compute the dot product of vectors, lengths of vectors, and angles between vectors.	1- Reinforced	Homework, Exams, or Quizzes
3. Compute the cross product of vectors and interpret it geometrically.	1- Reinforced	Homework, Exams, or Quizzes
4. Determine equations of lines and planes using vectors.	1- Reinforced	Homework, Exams, or Quizzes
5. Identify various quadric surfaces through their equations.	1- Reinforced	Homework, Exams, or Quizzes
6. Sketch various types of surfaces by hand and by using technology.	1- Reinforced	Homework, Exams, or Quizzes
7. Define vector functions of one real variable and sketch space curves.	1- Reinforced	Homework, Exams, or Quizzes
8. Compute derivatives and integrals of vector functions.	1- Reinforced	Homework, Exams, or Quizzes
9. Find the arc length and curvature of space curves.	1- Reinforced	Homework, Exams, or Quizzes
10. Find the velocity and acceleration of a particle moving along a space curve.	1- Reinforced	Homework, Exams, or Quizzes
11. Define functions of several variables and their limits.	1- Reinforced	Homework, Exams, or Quizzes
12. Calculate partial derivatives of functions of several variables.	1- Reinforced	Homework, Exams, or Quizzes
13. Interpret partial derivatives graphically.	1- Reinforced	Homework, Exams, or Quizzes
14. Apply the chain rule for functions of several variables.	1- Reinforced	Homework, Exams, or Quizzes
15. Calculate the gradient and directional derivatives of functions of several variables.	1- Reinforced	Homework, Exams, or Quizzes
16. Solve problems involving tangent planes and normal lines.	1- Reinforced	Homework, Exams, or Quizzes
17. Determine and classify the extrema of functions of several variables.	1- Reinforced	Homework, Exams, or Quizzes
18. Use the Lagrange multiplier method to find extrema of functions with constraints.	1- Reinforced	Homework, Exams, or Quizzes
19. Define double integrals over rectangles.	1- Reinforced	Homework, Exams, or Quizzes
20. Compute iterated integrals.	1- Reinforced	Homework, Exams, or Quizzes
21. Define and compute double integrals over general regions.	1- Reinforced	Homework, Exams, or Quizzes
22. Compute double integrals in polar coordinates.	1- Reinforced	Homework, Exams, or Quizzes
23. Compute triple integrals in Cartesian coordinates, cylindrical coordinates, and spherical coordinates.	1- Reinforced	Homework, Exams, or Quizzes
24. Apply triple integrals to find volumes.	1- Reinforced	Homework, Exams, or Quizzes
25. Sketch and interpret vector fields.	1- Reinforced	Homework, Exams, or Quizzes
26. Calculate line integrals along piecewise smooth paths, and interpret such quantities as work done by a force.	1- Reinforced	Homework, Exams, or Quizzes
27. Use the fundamental theorem of line integrals.	1- Reinforced	Homework, Exams, or Quizzes
28. Compute the curl and the divergence of vector fields.	1- Reinforced	Homework, Exams, or Quizzes
29. Determine whether a vector field is conservative.	1- Reinforced	Homework, Exams, or Quizzes
30. Compute surface integrals.	1- Reinforced	Homework, Exams, or Quizzes
31. Use double, triple and line integrals in applications, including Green's Theorem, Stokes' Theorem and the Divergence Theorem.	1- Reinforced	Homework, Exams, or Quizzes

3 Parametric Equations

Sometimes curves cannot be written as a function in one variable. For example, a circle fails the vertical line test. One way to deal with this is to write x and y in terms of another variable, usually __________. This is called a ____________________ representation or equation. For example, we can represent the points of the unit circle using parametric equations. The standard way to do this is to represent each point (x, y) on the circle by ____________________.



Example 3.1. *Sketch the curve described by the equations $x = 2 - 3t$, $y = 1 + t$. Then find a Cartesian equation of the curve.*



Example 3.2. *Find a parametric representation for the curve $y = x^2$.*

(a) $x(t) = t$ and $y(t) = t^2$
(b) $x(t) = -t$ and $y(t) = t^2$
(c) $x(t) = 1 - t$ and $y(t) = 1 - 2t + t^2$
(d) more than one of the above
(e) all of the above

Example 3.3. *Is $x(t) = t^2$ and $y(t) = t^4$ the same parametric equation as $x(t) = t$ and $y(t) = t^2$?*

(a) Yes and I am very confident.
(b)Yes, but I am not very confident.
(c) No and I am very confident.
(d) No, but I am not very confident.
(e) I love cats!

3.1 Calculus with Parametric Equations

Remember our beloved chain rule: The derivative of y with respect to t, $\frac{dy}{dt} =$

Thus $\dfrac{dy}{dx} =$

This formula allows us to find the derivative of a parametric curve without having to eliminate the parameter t.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{d}{dx} \cdot \frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example 3.4. *Find the equation of the tangent(s) to the curve $x = 1 + t^2$, $y = t + \frac{1}{t} + 3$ at $t = 1$.*

Objective 3: ***Area***
If we have a curve with parametric equations $x = f(t)$ and $y = g(t)$ for $\alpha \leq t \leq \beta$

the area under the curve is $A = \displaystyle\int_a^b y dx=$ or $\displaystyle\int_a^b x dy =$

Example 3.5. *Set up an integral that represents the area enclosed by the curve $x = 1+e^t$, $y = t-t^2$ and the x-axis.*

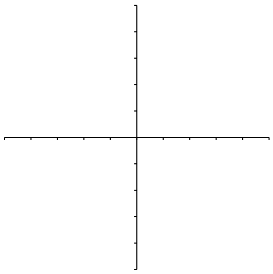
3.2 Arc Length

A curve with parametric equations $x = f(t)$ and $y = g(t)$ for $\alpha \leq t \leq \beta$ where f' and g' are continuous and $\dfrac{dy}{dx} > 0$ (which means our curve travels **once** from left to right as t increases from α to β) has an

Arc Length=

Example 3.6. *Set up an integral that represents the length of the curve $x = t + \sqrt{t}$, $y = t - \sqrt{t}$* $0.5 \leq t \leq 1$.

Example 3.7. *Find a Cartesian equation of the curve* $x = \sqrt{1+t}$, $y = \sqrt{t-1}$, *sketch the curve, and indicate with an arrow the direction in which the curve is traced as the parameter increases.*



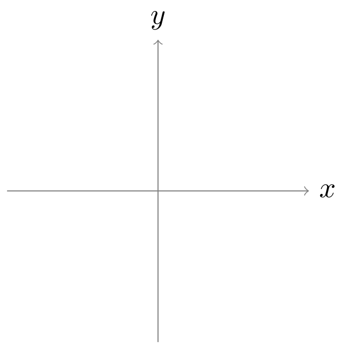
3.3 ICE Parametric Equations

1. a) Find the slope of the tangent line to $x = \ln(t)$, $y = \sqrt{t+1}$ at $t = 1$.

   b) Find the $\dfrac{d^2y}{dx^2}$

turn over for another question.

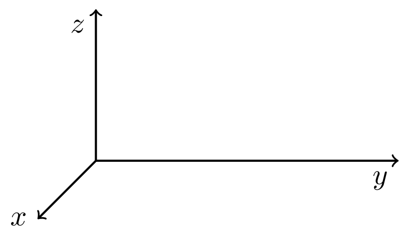
2. a) Sketch the curve described by $x = 2\sin t$, $y = 2\cos t$ (you can use a calculator).



b) Find the distance traveled by (x, y) as t varies over $0 \le t \le 3\pi$ for $x = 2\sin t$, $y = 2\cos t$. Compare it with the length of the curve.

4 3-Space

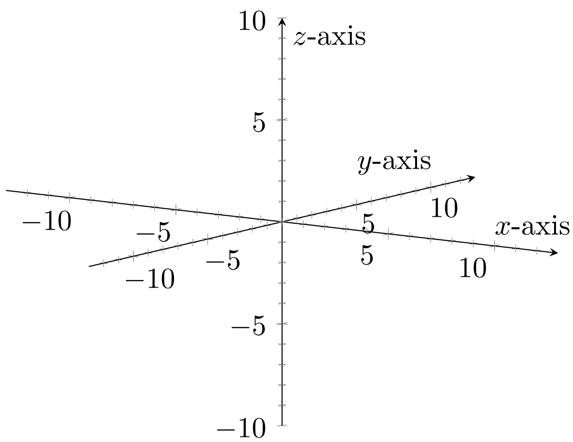
4.1 Cartesian Coordinates in 3 Space



Right Hand Rule:

Points will be in the form:

Labeling Axes is VERY Important!



The xy plane: the ____ coordinate is always

The xz plane: the ____ coordinate is always

The yz plane: the ____ coordinate is always

Example 4.1. *On the graph above, plot the points, the origin,* $Q = (-5, -5, 7)$*, and* $P = (3, 0, 5)$.

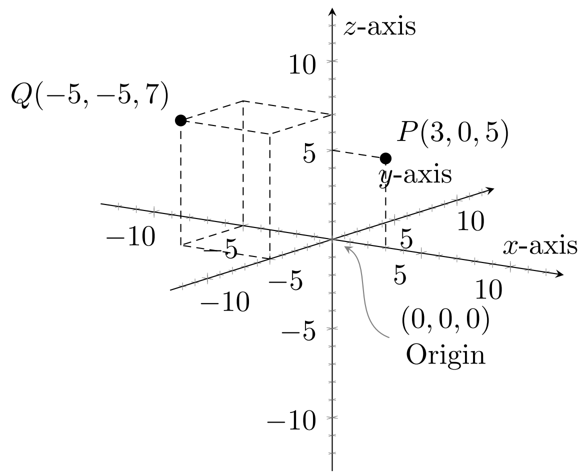
Two points determine a rectangular box called a ____________________. (See next page to see the parallelepiped created by Q and the origin.)

Distance Formula:

Midpoint Formula:

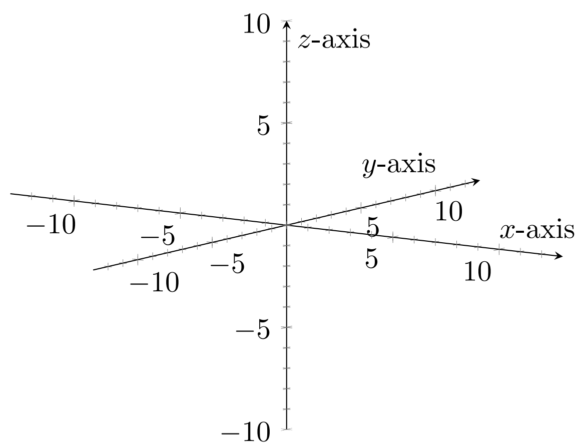
Equation of a Sphere:

Treat spheres like ellipses, but now we don't set one side equal to 1. Often we must __________ __________ __________ to find the center.



Example 4.2. *You awaken one morning to find that you have been transferred onto a grid which is set up like a standard right-hand coordinate system. You are at the point* $(-1, 3, -3)$*, standing upright, and facing the xz-plane. You walk 2 units forward, turn left, and walk for another 2 units. What is your final position?*

(a) (-1,1,-1)
(b) (-3,1,-3)
(c) (-3,5,-3)
(d) (1,1,-3)
(e) I am so turned around!



Example 4.3. *Which of the following points lies closest to the xy-plane?*

(a) (3,0,3)
(b) (0,4,2)
(c) (2,4,1)
(d) (2,3,4)

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

4.2 Linear Equations and Traces

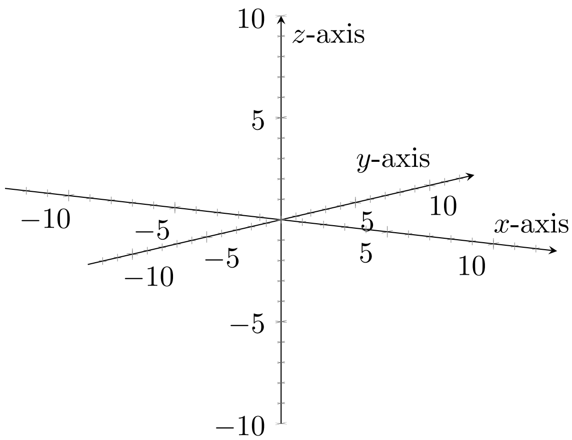
Equation of a Linear Equation is of the form:
where $A^2 + B^2 + C^2 \neq 0$ (aka A,B, and C are not all 0)

Fact: The graph of a linear equation is a ________.

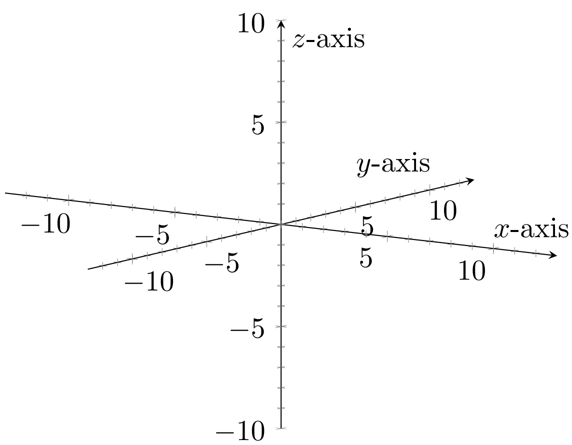
Method for sketching linear equations: Draw the Traces. Then Shade.

Definition: Traces are the line of intersection of a given plane with the coordinate axes.

Example 4.4. *Sketch the graph of* $6x - 3y + 15z = 3$



Example 4.5. *Sketch the graph of* $x = 4$



4.3 Parametric Equations in 3 Dimensions

We can easily extend parametric equations for 3 variables $x = f(t)$, $y = g(t)$, $z = h(t)$.

Definition: A curve is smooth if

Arc Length for 3-Space: For curve described by smooth curves, $x = f(t)$, $y = g(t)$, $z = h(t)$ is $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt$

5 Vectors

5.1 Introduction to Vectors

What is the difference between a vector and a scalar?

Examples of Scalars: Speed, work, energy, mass, length
Examples of Vectors: Velocity, force, torque, displacement

We can denote vectors with __________

We can denote vectors using components:

And we can denote vectors using **basis vectors**

Definition: Our *standard unit vectors* or *basis vectors* are
$\mathbf{i} =$ ____________________, $\mathbf{j} =$ ____________________, and $\mathbf{k} =$ ____________________

Definition: The **zero vector** is denoted by __________ and is the only vector without a

We can add or subtract vectors:

Triangle Law   Parallelogram Law   Component-wise

Warning: $< -3, 0, 5 >$ is different from $(-3, 0, 5)$!

And we can multiply a vector by a ____________________.

This is called "____________________ multiplication"

Warning: $< 1, 2, 5 > \cdot < 2, 7, -1 > \neq$

Example 5.1. *a) How does multiplying* $\mathbf{u}$ *by the scalar 2 change* $\mathbf{u}$*?*

b) How does multiplying $\mathbf{u}$ *by the scalar -1 change* $\mathbf{u}$*?*

c) What is $3\mathbf{u}$ *if* $\mathbf{u} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$*?*

d) Add $3\mathbf{u} - \mathbf{v}$ *if* $\mathbf{u} =< -1, 1, 2 >$*, and* $\mathbf{v} =< -3, 0, 5 >$*. Write the solution in terms of our basis vectors.*

Definition: The *length* or magnitude of a vector $\mathbf{u}$ is denoted __________ or __________. The magnitude of vector $\mathbf{u} =< u_1, u_2, u_3 >$ is calculated by determining the distance between the point at the head of the vector, (u_1, u_2, u_3) and the origin.
Thus $|\mathbf{u}| =$

Definition A vector having length one is called a __________ vector.

Properties of Vectors

Most of our basic rules apply to vectors.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u}$
3. $a(b\mathbf{u}) = (ab)\mathbf{u}$
4. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
5. $||a\mathbf{u}|| = |a\mathbf{u}| = |a||\mathbf{u}|$
6. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
7. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $1\mathbf{u} = \mathbf{u}$

Property: Two vectors are parallel if they point in the same direction. That means $\mathbf{u}$ and $\mathbf{v}$ are parallel if there exists a constant c such that

Example 5.2. *a) Using you answer from the previous example, what is* $|3\mathbf{u} - \mathbf{v}|$*?*

b) Find a unit vector with the same direction as $3\mathbf{u} - \mathbf{v}$*.*

5.2 Displacement Vectors

Definition: The **position vector** of a point (a_1, a_2, a_3), is the vector $< a_1, a_2, a_3 >$ which is the vector that starts at the origin and ends at the point (a_1, a_2, a_3).

Definition: The **displacement vector** of 2 points $U = (u_1, u_2, u_3)$ and $V = (v_1, v_2, v_3)$ is the vector that represents the shortest distance from U to V. In this case our displacement vector for U and V is

Example 5.3. *A cat is sitting on the ground at the point* $(1, 4, 0)$ *watching a squirrel at the top of the tree. The tree is one unit high and its base is the point* $(2, 4, 0)$*. Find the displacement vectors for the following:*
a) The "origin" to the cat.

b) The bottom of the tree to the squirrel.

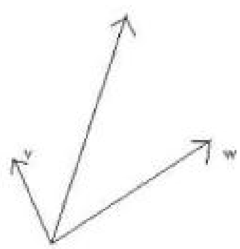
c) The bottom of the tree to the cat.

d) From the cat to the squirrel.

Example 5.4. *An airplane flies horizontally from west to east at 200 mph relative to the air. If it is in a steady 40 mph wind that blows southeast (45° south of east). Find the speed and direction of the plane relative to the ground.*

5.3 ICE Vectors

1. In the picture below, the unlabeled vector is closest to



(a) $\mathbf{v} + \mathbf{w}$
(b) $\mathbf{v} \quad \mathbf{w}$
(c) $\mathbf{v} + 2\mathbf{w}$
(d) $2\mathbf{v} + \mathbf{w}$
(e) I want to go home

2. True or False: The vector $< \frac{1}{2}, \frac{1}{2} >$ is a unit vector.
(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3. Find a vector that points in the same direction as $< 2, 1, 2 >$, but has a magnitude of 5.
a) $< \frac{10}{3}, \frac{5}{3}, \frac{10}{3} >$
b) $< \frac{10}{\sqrt{3}}, \frac{5}{\sqrt{3}}, \frac{10}{\sqrt{3}} >$
c) $< \frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}} >$
d) $< 10, 5, 10 >$
e) $< 30, 15, 30 >$
f) More than one of the above

Turn over!

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

4. The vector $\mathbf{F}$ represents the force exerted on an object. It has magnitude $|\mathbf{F}|$ and a direction given by θ. We denote the horizontal and vertical components of $\mathbf{F}$ as F_x and F_y respectively. Then $F_x = |\mathbf{F}| \cos\theta$ and $F_y = |\mathbf{F}| \sin\theta$ and the force vector is $\mathbf{F} =< F_x, F_y >$.

   Suppose you pull a suitcase with a strap that makes a 60° angle with the horizontal. The magnitude of the force you exert on the suitcase is 40 lbs.

   a) Find the horizontal and vertical components of the force.

   b) Is the horizontal component of the force greater if the angle of the strap is 45° instead of 60°?

   c) Is the vertical component of the force greater if the angle of the strap is 45° instead of 60°?

5. A plane is heading 60° north of east at an airspeed of 700 km/hr, but there is a wind blowing from the west at 50 km/hr. What is the plane's speed relative to the ground? In what direction does the plane end up flying?

5.4 Dot Products

So far we can add and subtract vectors and multiply them by scalars. Today we will discuss a "multiplication" for two vectors called the __________ product, denoted by __________

We use dot products to determine angles between vectors and projections.

Definition 1 of Dot Product: For $\mathbf{u} =< u_1, u_2, u_3 >$ & $\mathbf{v} =< v_1, v_2, v_3 >$,
$\mathbf{u} \cdot \mathbf{v} =$

Sometimes we call the dot product the __________ product. Why?

Properties of Dot Products

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot \mathbf{0} = \mathbf{0}$
3. $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v}$
4. $a(\mathbf{u} \cdot \mathbf{v}) = (a\mathbf{u}) \cdot \mathbf{v}$
5. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

Definition 2 of Dot Product: For nonzero vectors $\mathbf{u}$ & $\mathbf{v}$, $\mathbf{u} \cdot \mathbf{v} =$

where θ is the smallest non-negative angle between $\mathbf{u}$ & $\mathbf{v}$. That is, $0 \leq \theta \leq \pi$.

Notice that if $\mathbf{u} = 0$ or $\mathbf{v} = 0$, then $\mathbf{u} \cdot \mathbf{v} =$ __________ and θ is ____________________.

Note: This definition is useful for determining angles between vectors.

Example 5.5. *Let* $\mathbf{u} =< 2, 7, 9 >$, $\mathbf{v} =< -1, 7, 0 >$.

a) What is $\mathbf{u} \cdot \mathbf{v}$?

b) What is $\mathbf{v} \cdot \mathbf{v}$?

Example 5.6. *The only way that* $\mathbf{u} \cdot \mathbf{v} = 0$ *is if* $\mathbf{u} = \mathbf{0}$ *or* $\mathbf{v} = \mathbf{0}$.
(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
(e) I love Calculus!

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Definition: Two vectors $\mathbf{u}$ & $\mathbf{v}$ are said to be ____________________ if $\mathbf{u} \cdot \mathbf{v} =$ ____________.
That is, $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

Definition: A vector that is perpendicular to a plane is called a __________ vector for the plane.

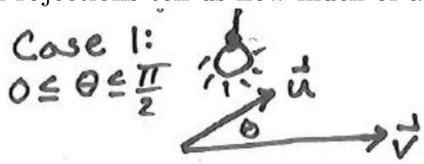
Example 5.7. *The zero vector,* $\mathbf{0}$ *is orthogonal to all other vectors.*
(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
(e) I love cats!

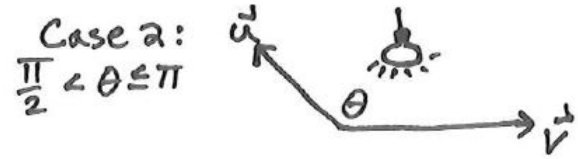
Example 5.8. *Find the angle made by* ABC *where* $A = (1, -3, -2)$*,* $B = (2, 0, -4)$*, and* $C = (2, 1, -4)$*.*

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

5.4.1 Projections

Projections tell us how much of a given vector lies in the direction of another vector.





Definition: The *orthogonal projection* of $\mathbf{u}$ onto $\mathbf{v}$ where $\mathbf{v} \neq 0$ is $\text{Proj}_{\mathbf{v}}\mathbf{u}$

Which part denotes the length of the projection?
Which part denotes the direction?

Definition: The *scalar projection* of $\mathbf{u}$ onto $\mathbf{v}$ where $\mathbf{v} \neq 0$ is $\text{Scal}_{\mathbf{v}}\mathbf{u} =$
Using our properties of dot products, we see that $\text{Scal}_{\mathbf{v}}\mathbf{u} =$

Some texts called the scalar projection the *scalar component or* $\mathbf{u}$ *in the direction of* $\mathbf{v}$.

Now we have another definition for $\text{Proj}_{\mathbf{v}}\mathbf{u}$

Definition: The work done by a constant force $\mathbf{F}$ in moving an object along the line from A to B is the magnitude of the force in the direction of the motion multiplied by the distance moved. If $\mathbf{D}$ is the displacement vector from A to B, then the work done is __________.

That is, the work is $\text{Scal}_{\mathbf{D}}\mathbf{F}||\mathbf{D}|| =$

Example 5.9. *Given Let* $\mathbf{u} =< 2, 7, 9 >$, $\mathbf{v} =< -1, 7, 0 >$ *as in Example 5.5, what is Proj*$_{\mathbf{v}}\mathbf{u}$*? (Hint: Use your answers from Example 5.5!)*

5.5 Cross Products

Last class we talked about the scalar product of two vectors. Today we will discuss the vector product. For this product when we "multiply" two vectors we get a __________ !!!!!!!!!!!!!

We use cross products to determine normal vectors, torque, magnetic force, and well yeah we gotta know how to take cross products.

Definition 1 of Cross Product: For nonzero vectors $\mathbf{u}$ & $\mathbf{v}$, $\mathbf{u} \times \mathbf{v} =$
where $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{u}$ & $\mathbf{v}$ and whose direction is determined by the __________ __________ __________.

Note: $||\mathbf{u} \times \mathbf{v}|| =$

Area of a Parallelogram: The area of the parallelogram made from the 2 sides of $\mathbf{u}$ & $\mathbf{v}$ is __________

Volume of a Parallelepiped: The volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is

Definition 2 of Cross Product: For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ & $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$,
$\mathbf{u} \times \mathbf{v} =$

Properties of Cross Products

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
3. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
4. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
5. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$
6. $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

Provable Facts:

$\mathbf{i} \times \mathbf{j} =$ $\qquad$ $\mathbf{j} \times \mathbf{k} =$ $\qquad$ $\mathbf{k} \times \mathbf{i} =$

Example 5.10. *Confirm that* $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

Example 5.11. *Find a vector normal to* $\mathbf{u} =< 1, 2, 3 >$ *&* $\mathbf{v} =< 2, -5, -3 >$.

Theorem 1: For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ & $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$ are both equal to __________

"Proof":

Theorem 2: $\mathbf{u}$ & $\mathbf{v}$ in a 3-Space are parallel if and only if $\mathbf{u} \times \mathbf{v} =$ __________

Notice this is easy to see from our angle definition of the cross product since two vectors are parallel if and only if the angle between them is either $0°$ or $180°$ which means $\sin\theta =$ __________.

5.5.1 Torque

When we are using a wrench, the "twisting" power you generate depends on 3 things

1.
2.
3.

Definition: *Torque* is the a twisting generated by a force acting at a distance from a pivot point. If a force $\mathbf{F}$ is applied to a point P at the head of a vector $\mathbf{r} = \vec{OP}$ the torque, $\vec{\tau} =$ __________

$||\vec{\tau}|| =$
where θ is the angle between $\mathbf{r}$ & $\mathbf{F}$.

Example 5.12. *Let* $\mathbf{r} = \vec{OP} =< 1, -1, 2 >$ *and suppose a force* $\mathbf{F} =< 10, 10, 0 >$ *is applied to P. Find the torque about O that is produced.*

5.6 ICE Dot and Cross Products

Cross and Dot products

1. For the following questions, consider the vectors $\mathbf{u} = 2\mathbf{i}$ and $\mathbf{v} = 3\mathbf{j}$

   a) Draw the two vectors.

   b) Find the magnitude of $\mathbf{u} \times \mathbf{v}$.

   c) Determine the direction of $\mathbf{u} \times \mathbf{v}$ without any calculations.

   d) Verify your previous answers by calculating $\mathbf{u} \times \mathbf{v}$ in component form.

   e) Find the magnitude of $\mathbf{v} \times \mathbf{u}$

   f) Determine the direction of $\mathbf{v} \times \mathbf{u}$ without any calculations.

   g) Verify your previous answer by calculating $\mathbf{v} \times \mathbf{u}$ in component form.

2. Two vectors have a dot product of 14. To guarantee the dot product is equal to 28, you could:
   (a) double the angle between the vectors
   (b) double the length of both vectors
   (c) double the length of one vector
   (d) none of the above
   (e) Archer is my favorite!

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

3. When vectors are closely aligned, is $\text{Scal}_{\mathbf{v}}\mathbf{u}$ large or small?
   (a) Large, and I am very confident
   (b) Large, but I am not very confident
   (c) Small, but I am not very confident
   (d) Small, and I am very confident
   (e) I want this class to never end!

4. What is $\text{Proj}_{\mathbf{v}}\mathbf{u}$ when $\mathbf{v}$ & $\mathbf{u}$ are orthogonal?
   (a) $\mathbf{v}$
   (b) undefined
   (c) $\mathbf{0}$
   (d) $-\mathbf{v}$
   (e) 0

5. A vector that is normal to the plane containing the vectors $\mathbf{a} = 4\mathbf{i}$ -$\mathbf{j}$ + $2\mathbf{k}$ and $\vec{b}$ =-$\mathbf{i}$ + $5\mathbf{j}$ + $\mathbf{3k}$ is
   (a) -13$\mathbf{i}$ + 14$\mathbf{j}$ + 1$\mathbf{9k}$
   (b) 13$\mathbf{i}$ + 14$\mathbf{j}$ - 19$\mathbf{k}$
   (c) -13$\mathbf{i}$ - 14$\mathbf{j}$ + 19$\mathbf{k}$
   (d) 13$\mathbf{i}$ - 14$\mathbf{j}$ -19$\mathbf{k}$
   (e) More than one of the above.

6. If $\vec{d} = \vec{a} \times \vec{b}$, then $\vec{a} \cdot \vec{d} =$
   (a) $\vec{a} \times (\vec{b} \cdot \vec{b})$
   (b) $\mathbf{0}$
   (c) $\vec{a} \times (\vec{a} \cdot \vec{b})$
   (d) $(\vec{a} \cdot \vec{b}) \times \vec{b}$
   (e) none of the above.

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

6 Vector Valued Functions

Definition: A vector-valued function $\mathbf{F}(t)$ associates each t with a vector:
$\mathbf{F}(t) =$

Note: Now there is a difference now between $\mathbf{f}(t)$ and $f(t)$

Since $\mathbf{F}(t)$ is a vector, we can do what operations to vector valued functions?

So now we just now have functions of t rather than just numbers.

And, since $\mathbf{F}(t)$ is a function, we also want to discuss calculus ideas in terms of $\mathbf{F}(t)$.

Calculus for Vector Valued Functions: For $\mathbf{F}(t) =< f(t), g(t), h(t) >$,

$\mathbf{F}'(t) =$

$\int \mathbf{F}(t)dt =$

$\lim\limits_{t\to a} \mathbf{F}(t) =$

$\frac{d}{dt}[\mathbf{F}(t) + \mathbf{G}(t)] =$

$\frac{d}{dt}[c\mathbf{F}(t)] =$

$\frac{d}{dt}[p(t)\mathbf{F}(t)] =$

$\frac{d}{dt}[\mathbf{F}(t) \cdot \mathbf{G}(t)] =$

$\frac{d}{dt}[\mathbf{F}(t) \times \mathbf{G}(t)] =$

$\frac{d}{dt}[\mathbf{F}(p(t))] =$

What about a quotient rule?

Example 6.1. *Let* $\mathbf{a}(t) =< 2t, 0, e^{-t} >$ *and* $\mathbf{b}(t) =< \cos t, 5t, 1 >$. *Determine* $\frac{d}{dt}[\mathbf{a}(t) \cdot \mathbf{b}(t)]$.

Example 6.2. *Could I have done the dot product first and then differentiated term by term?*

a) Yes and I am very confident
b) Yes, but I am not very confident
c) No and I am very confident
d) No, but I am not very confident
e) I love cats.

Example 6.3. *Does order matter in the derivative of the dot product?*

a) Yes and I am very confident
b) Yes, but I am not very confident
c) No and I am very confident
d) No, but I am not very confident
e) I hate math.

Example 6.4. *Does order matter in the derivative of the cross product?*

a) Yes and I am very confident
b) Yes, but I am not very confident
c) No and I am very confident
d) No, but I am not very confident
e) Will this class ever end?

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Example 6.5. *Let* $\mathbf{a}(t) =< 2t, 0, e^{-t} >$ *and* $f(t) = 7t$. *Determine* $\frac{d}{dt}[f(t)\mathbf{a}(t)]$.

Example 6.6. *Let* $\mathbf{a}(t) =< 2t, 0, e^{-t} >$ *and* $\mathbf{b}(t) =< \cos t, 5t, 1 >$. *Determine* $\frac{d}{dt}[\mathbf{a}(t) \times \mathbf{b}(t)]$.

Definition: Given a position vector $\mathbf{r}(t) =< f(t), g(t), h(t) >$,

the velocity vector $\mathbf{v}(t) =$

the acceleration vector $\mathbf{a}(t) =$

And the speed is given by

Example 6.7. *Consider the trajectory given by the position vector* $\mathbf{p}(t) =< e^{-t}\cos t, e^{-t}\sin t, 2-2e^{-t} >$*. for* $t \geq 0$.

a) What is the initial point of the trajectory?

b) What is the terminal point of the trajectory?

c) What is the initial velocity? What is the initial speed?

Just for The Cool Cats: When is the speed the greatest? This is long, but interesting. =)

7 Lines and Planes

7.1 Lines In Space!

What determines a line?

Representation of a Line: Let $\vec{v} =< a, b, c >$ be a direction vector for a line and let $\vec{r_0} =< x_0, y_0, z_0 >$ be the position vector for a point $P_0 = (x_0, y_0, z_0)$ on the same line.

Vector Representation of a Line: $\vec{r} =$

Parametric Representation of a Line:
$x =$
$y =$
$z =$

Symmetric Representation of a Line:

Note: If a, b, or $c = 0$, we can still eliminate t. For example if $b = 0$, then we have

Example 7.1. *Find an equation of a line that contains* $u = (2, 7, 9)$ *and is parallel to the vector* $< 3, -1, 7 >$.

Example 7.2. *Find an equation of a line that contains* $u = (2, 7, 9)$ *and* $v = (1, -1, 1)$.

Example 7.3. *Find the symmetric equation of the line through* $(-5, 7, -3)$ *that is perpendicular to both* $< 2, 1, -3 >$ *and* $< 5, 4, -1 >$.

7.2 Planes

We already mentioned that an equation of a plane is in the form $Ax + By + Cz = D$. Now we will discuss another way to describe a plane.

Recall a line in the xy plane can be described with a __________ and a __________.

A plane can be described by a __________ and a __________ vector.

Standard Form of Plane: The equation of the plane going through the point $P = (x_1, y_1, z_1)$ with a normal vector $\mathbf{n} =< a, b, c >$ can be written as

Useful fact: Planes $Ax + By + Cz = D$ and $ax + by + cz = d$ are parallel if there exists a constant k such that $< A, B, C >= k < a, b, c >$. The planes are identical if they are parallel and $D = kd$.

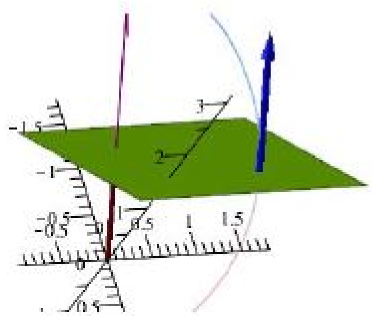
Example 7.4. *a) Find the equation of the plane through (0,0,1) and perpendicular to* $2\mathbf{i}+4\mathbf{j}+\mathbf{k}$.

b) Find the equation of the plane through $(-1, 2, 5)$ *and parallel to the plane you found in part a.*

7.3 Tangent Lines

If a curve, $\mathbf{r}$ has the position vector $< f(t), g(t), h(t) >$, the tangent line to $\mathbf{r}$ has the direction vector:

There is *exactly* __________ plane perpendicular to a smooth curve at any given point. If we have a direction vector for the tangent line to the curve at P, then it is a __________ vector for the plane.



Example 7.5. *Find the equation of the plane perpendicular to the curve* $\mathbf{r}(t) =< \cos(t), \sin(t), t >$ *at* $t = \dfrac{\pi}{2}$.

7.4 ICE Vector Functions

Consider the curve traced by the vector valued function $\mathbf{r}(t) =< \sin t \cos t, \cos^2 t, \sin t >$.

1. Show that the curve lives on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

2. Compute $|\mathbf{r}(t)|$. [Hint: this should now be easy!]

3. Compute $\mathbf{r}'(t)$.

4. Show that $\mathbf{r}'(t)$ is perpendicular to $\mathbf{r}(t)$ for all t. (You can verify this analytically. Also this is a general result, true whenever $|\mathbf{r}(t)|$ is constant.) You can verify analytically on the back of the page after completing 5 and 6.

5. Find the tangent line to the curve at $t = 0$.

6. If a rocket were flying along the curve traced by $\mathbf{r}(t)$ and suddenly turned off its engines at $t = 0$ where would it be when $t = 2$?

7. Verify analytically that $\mathbf{r}'(t)$ is perpendicular to $\mathbf{r}(t)$ for all t.

8 Curvature

8.1 Definitions of Curvature

There are 2 ways to change velocity. We can change our __________ or change our __________. Intuitively, curvature measures how much a curve bends at a point.

Given a position vector, $\vec{r}$, what is our tangent vector?
Let $\mathbf{T}(t)$ represent the unit tangent vector for $\vec{r}$. So $\mathbf{T}(t) =$

So we can define curvature as $\kappa = \|\frac{d\mathbf{T}}{ds}|$

Note we are differentiating with respect to s which represents __________ __________. This definition is not very helpful though. Instead we can do some manipulation (see text) and arrive at the following definition for curvature:

Definition 1: The curvature, κ, of curve $\mathbf{r}(\mathbf{t})$ is $\kappa(t) =$

Alternative Definition 2: $\kappa(t) =$

Alternative Definition 3: Given $\mathbf{r}(t) =< f(t), g(t) >$, $\kappa(t) =$

Example 8.1. *Find the curvature of* $\mathbf{r}(t) =< t, \ln(\cos(t)) >$ *using Definition 1.*

Example 8.2. *Find the curvature of* $\mathbf{r}(t) =< t, \ln(\cos(t)) >$ *using Definition 2.*

Example 8.3. *Find the curvature of* $\mathbf{r}(t) =< t, \ln(\cos(t)) >$ *using Definition 3.*

Solution: $f'(t) = 1, f''(t) = 0$. $g'(t) = \dfrac{-\sin(t)}{\cos(t)} = -\tan(t)$, $g''(t) = -\sec^2(t)$. Thus

$$\kappa(t) = \frac{|f' \cdot g'' - g' \cdot f''|}{((f')^2 + (g')^2)^{\frac{3}{2}}} = \frac{|1 \cdot (-\sec^2(t)) - (-\tan(t)) \cdot 0|}{(1^2 + (-\tan(t))^2)^{\frac{3}{2}}} = \frac{sec^2(t)}{(\sec^2(t))^{\frac{3}{2}}} = \frac{\sec^2(t)}{\sec^3(t)} = \cos(t)$$

8.2 Components of Acceleration

Let $\mathbf{T}(t)$ represents the unit tangent vector for $\vec{r}$.
Thus $\mathbf{T}(t) \cdot \mathbf{T}(t) =$ ________ and thus

So $\mathbf{T}(t) \cdot \mathbf{T}'(t) =$ ________ $\Rightarrow \mathbf{T}(t)$ is ________ with $\mathbf{T}'(t)$.

Definition: We define this normal vector (after normalizing it) as the unit normal vector $\mathbf{N}(t) =$

Note that there are infinitely many unit vectors perpendicular to $\mathbf{T}$ at a point. $\mathbf{N}(t)$ is the principle unit normal vector and is normal to $\mathbf{T}(t)$ and points in the direction of "curving".

One other important unit vector is our binormal vector which is orthogonal to both $\mathbf{T}$ & $\mathbf{N}$.

Definition: The unit binormal vector $\mathbf{B}(t) =$

Example 8.4. *Determine the unit vectors* $\mathbf{T}$, $\mathbf{N}$ *and* $\mathbf{B}$ *for* $\mathbf{r}(t) =< t, \ln(\cos(t)) >$.

When we are in a car which is accelerating forward, we feel pushed __________ and if we turn right, we feel pushed to the __________. These two kinds of acceleration are called the *tangential and normal components of acceleration.*

We can write our $\mathbf{a}(t)$ in terms of these components. That is, we can write $\mathbf{a}(t)$=

We can determine the scalars a_T and a_N by:
Definitions of Components:

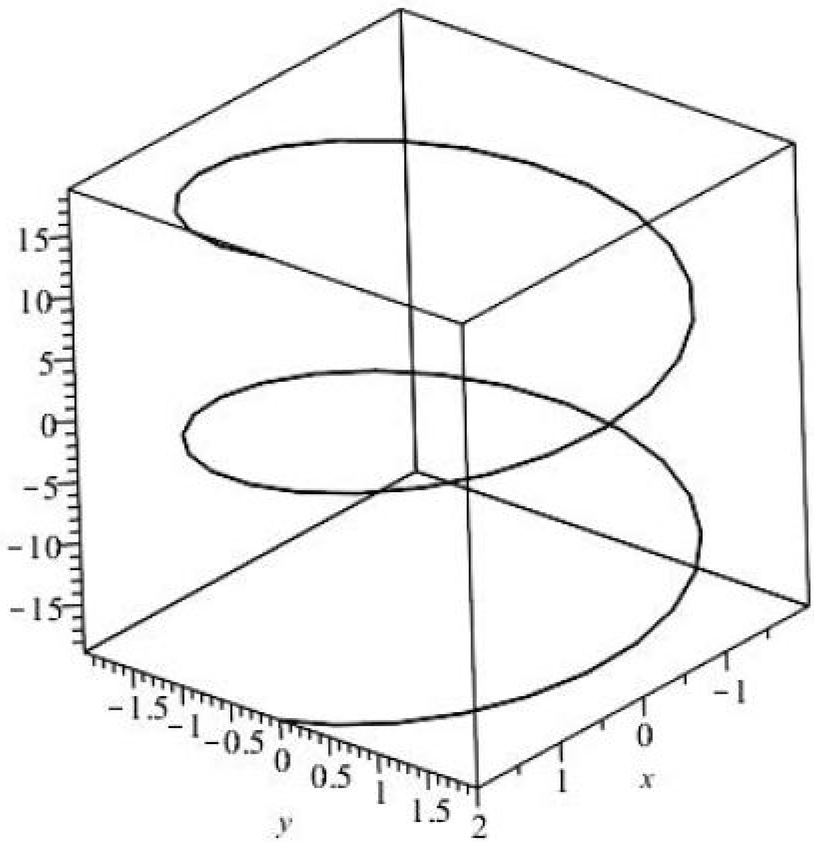
$a_t =$ $\qquad\qquad$ $a_N =$

Example 8.5. *Determine the components of acceleration for* $\mathbf{r}(t) =< t, \ln(\cos(t)) >$ *at* $t = 0$.

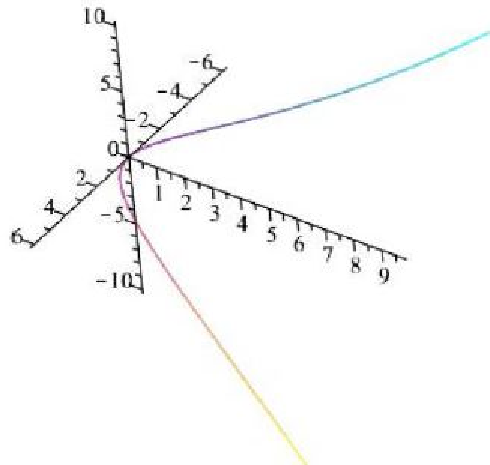
8.3 ICE Curvature

2 sides!

1. The curve $\mathbf{r}(t) =< 2\cos t, 2\sin t, 3t >$ is shown below. Find $\mathbf{T}, \mathbf{N}$, & $\mathbf{B}$. Sketch $\mathbf{T}, \mathbf{N}$, & $\mathbf{B}$ when $t = \dfrac{3\pi}{2}$.



2. Below is the graph of the space curve given by the vector valued equation
$\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} - \frac{1}{3}t^3\mathbf{k}$.



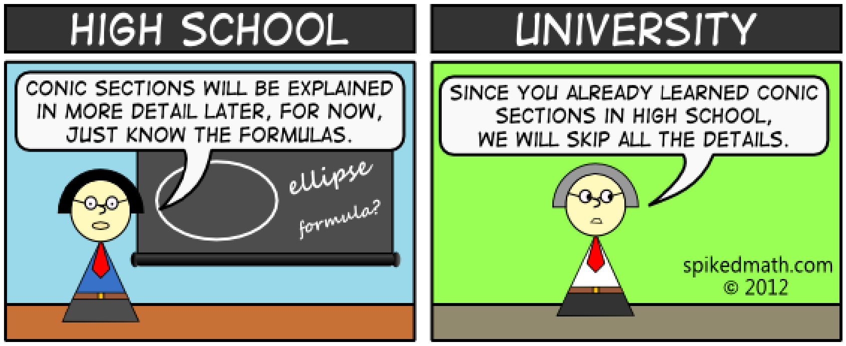
a) Based on an inspection of the graph, at what point is the curvature the greatest?

b) Find the curvature function $\kappa(t)$ for the space curve. Factoring and using some algebra will lead you to a "nice" answer which will help you answer part (c) of the question.

c) Verify analytically the point at which the curvature is the greatest. You may need to do some factoring or algebra to make your answer from part b nicer to analyze.

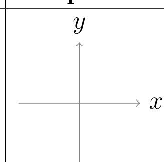
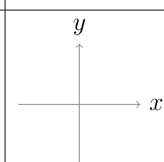
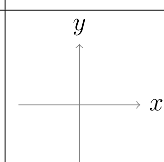
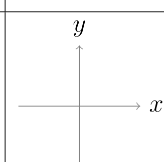
9 Surfaces

9.1 Review of Conic Sections

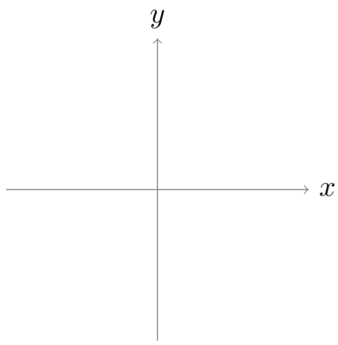


9.1.1 Parabolas

What is a parabola?

Parabola Notes						
Function	**Equation**	**Vertex**	**Focus**	**Directrix**	**Opens**	**Graph**
Standard Vertical	$x^2 = 4py$	$(0, 0)$	$(0, p)$	$y = -p$	up if $p > 0$ down if $p < 0$	
General Vertical	$(x - h)^2 = 4p(y - k)$	(h, k)	$(h, k + p)$	$y = k - p$	up if $p > 0$ down if $p < 0$	
Standard Horizontal	$y^2 = 4px$	$(0, 0)$	$(p, 0)$	$x = -p$	right if $p > 0$ left if $p < 0$	
General Horizontal	$(y - k)^2 = 4p(x - h)$	(h, k)	$(h + p, k)$	$x = h - p$	right if $p > 0$ left if $p < 0$	

Example 9.1. *Sketch the graph of* $x^2 = -8y$.



Example 9.2. *Find an equation of the parabola that has vertex (2, 1) and directrix $y = 5$.*

(a) $(x-2)^2 = 20(y-1)$

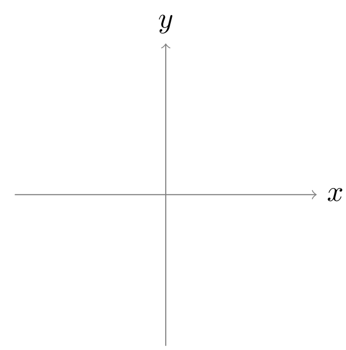
(b) $(x-2)^2 = -20(y-1)$

(c) $(y-1)^2 = 20(x-2)$

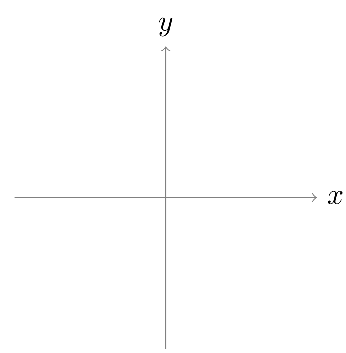
(d) $(y-1)^2 = -20(x-2)$

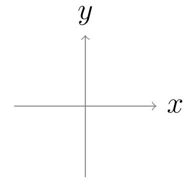
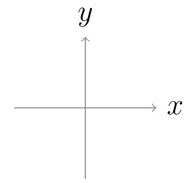
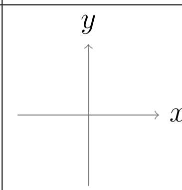
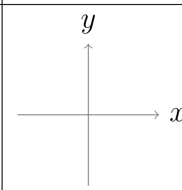
9.1.2 Ellipses and Hyperbolas

What is an ellipse?



What is a hyperbola?



Ellipse Notes					
Function	**Equation**	**Center**	**Foci**	**Vertices**	**Graph**
Standard Horizontal (Wide)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $a > b$ $c^2 = a^2 - b^2$	$(0, 0)$	$(\pm c, 0)$	major:$(\pm a, 0)$ minor:$(0, \pm b)$	
General Horizontal (Wide)	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ $a > b$ $c^2 = a^2 - b^2$	(h, k)	$(h \pm c, k)$	minor: $(h, k \pm b)$ major: $(h \pm a, k)$	
Standard Vertical (Tall)	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$ $a > b$ $c^2 = a^2 - b^2$	$(0, 0)$	$(0, \pm c)$	minor:$(\pm b, 0)$ major:$(0, \pm a)$	
General Vertical (Tall)	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$ $a > b$ $c^2 = a^2 - b^2$	(h, k)	$(h, k \pm c)$	minor:$(h \pm b, k)$ major:$(h, k \pm a)$	

Given a conic sections of the form: $Ax^2 + Cy^2 + Dx + Ey + F = 0$, we have a...

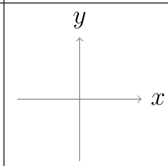
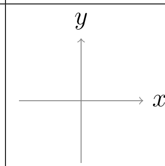
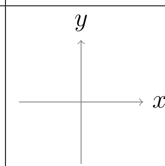
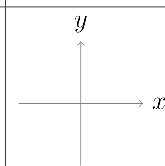
Circle if:

Hyperbola if:

Ellipse if:

Parabola if:

Example 9.3. *What type of conic section is* $4x^2 - 9y^2 + 16x + 18y = 29$*?*

Hyperbola Notes						
Function	**Equation**	**Center**	**Foci**	**Vertices**	**Asymptotes**	**Graph**
Standard Vertical (Tall)	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ $c^2 = a^2 + b^2$	$(0,0)$	$(0, \pm c)$	$(0, \pm b)$ $(\pm a, 0)$	$y = \pm \frac{a}{b} x$	
General Vertical (Tall)	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ $c^2 = a^2 + b^2$	(h,k)	$(h, k \pm c)$	$(h, k \pm b)$ $(h \pm a, k)$	$y = \pm \frac{a}{b}(x-h) + k$	
Standard Horizontal (Wide)	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $c^2 = a^2 + b^2$	$(0,0)$	$(\pm c, 0)$	$(\pm a, 0)$	$y = \pm \frac{b}{a} x$	
General Horizontal (Wide)	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $c^2 = a^2 + b^2$	(h,k)	$(h \pm c, k)$	$(h \pm a, k)$	$y = \pm \frac{b}{a}(x-h) + k$	

Another way to define these conic sections is by a fixed positive constant e called the *eccentricity* for all points P,

$$e = \frac{|PF|}{Pl}$$

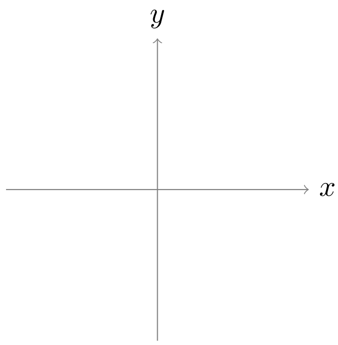
Where F is a fixed point called the *focus* and l is a fixed line called the *directrix* in a plane.

We have an ellipse if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$

Example 9.4. *Find the vertices/major axis and the endpoints of the minor axis for the ellipse given by the equation* $9x^2 + 4y^2 = 16$.

(a) vertices: $(\pm 2, 0)$; endpoints of minor axis: $(0, \pm 4/3)$
(b) vertices: $(0, \pm 2)$; endpoints of minor axis: $(\pm 4/3, 0)$
(c) vertices: $(\pm 2, 0)$; endpoints of minor axis: $(0, \pm 3/4)$
(d) vertices: $(0, \pm 2)$; endpoints of minor axis: $(\pm 3/4, 0)$

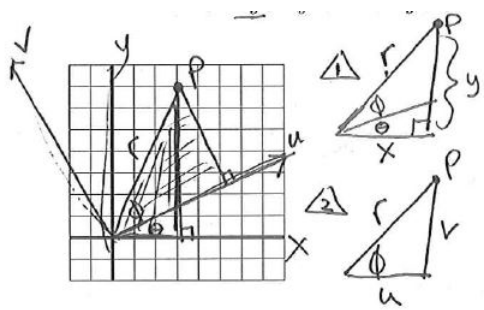
Example 9.5. *Sketch the graph of* $y^2 - 4x^2 - 2y + 16x = -1$.



9.1.3 Optional Topic: Translations and Rotations

We know how to deal with Conic Sections in the form: $Ax^2 + Cy^2 + Dx + Ey + F = 0$.
But what if we have an xy term like $4x^2 - 3xy = 18$?
In Example 3, we wrote $y^2 - 4x^2 - 2y + 16x = -1$ as $\dfrac{(x+2)^2}{4} - \dfrac{(y-1)^2}{16} = 1$ by completing the square.
We can think of this as a __________ of axes by setting new coordinates $u =$ __________ and $v =$ __________. The u-v axes are __________ and __________ to the x-y axes.

If we have $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ as our form, this is a __________ of the x-y axes.



By Triangle 1: $\cos(\theta + \phi) = \dfrac{x}{r}$
$x = r\cos(\theta + \phi) = r(\cos\theta\cos\phi - r\sin\theta\sin\phi)$
$= (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta$

By Triangle 2: $\cos\phi = \frac{u}{r} \Rightarrow u = r\cos\phi$
$\sin\phi = \frac{v}{r} \Rightarrow v = r\sin\phi$

Thus
$x =$
$y =$

How do we determine the angle θ?

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

$\cot(2\theta) =$ __________
Then we just solve for θ!

Example 9.6. *Eliminate the cross-product term for* $2xy - 1 = 0$ *and put it in terms of* u *and* v *to recognize the conic section.*

9.2 Graphing Surfaces

Today, we will work on graphing three dimensional surfaces. When graphing surfaces in a 3-Space, it often helpful to look at good cross sections -especially traces!

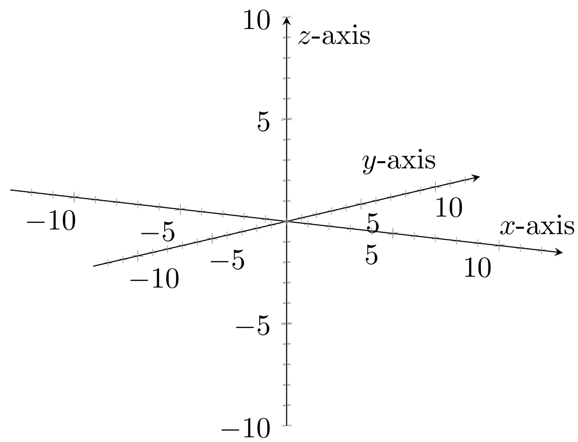
Definition: A *cross section* of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes.

__________ are cross sections of the xy,xz, and yz planes.

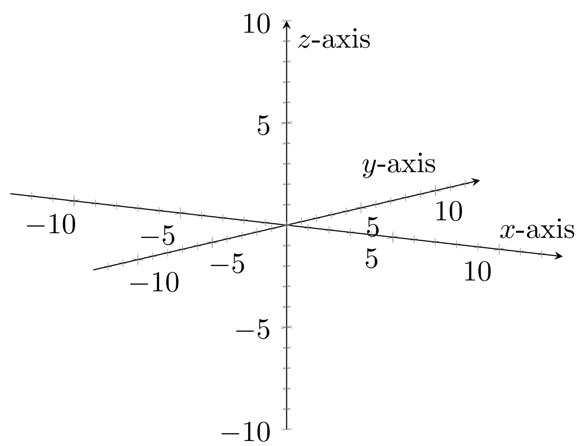
It may also be useful to memorize basic surfaces. But even if you do this, it is good to know how to check if you have the correct curve. Let's start with some basic examples of surfaces.

Example 9.7. *Provide a rough sketch and describe in words the surfaces described by the following equations. Key: Think abut the surfaces and traces before you calculate anything.*

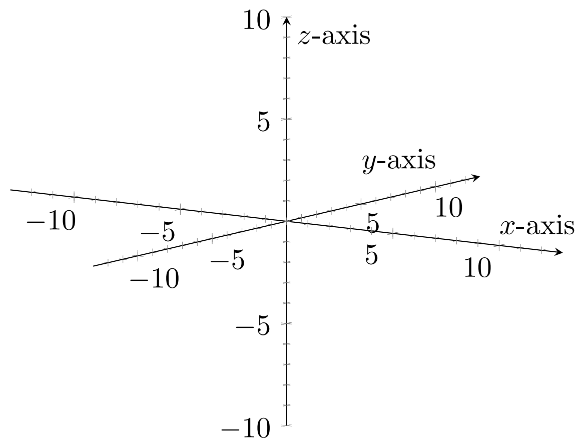
a) $x^2 + y^2 + z^2 = 25$



b) $x^2 + y^2 = 3$



c) $y^2 + z^2 = 1$



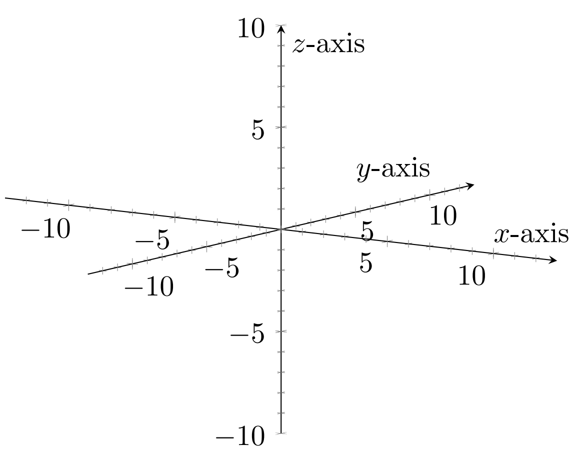
Notice in Example 9.7 b and c, we were missing one of our three variables. These are examples of a family of functions called ____________.

Definition: Given a curve C in a plane P and a line l not in P, a *cylinder* is the surface consisting of all lines parallel to l that pass through C.

Steps for Sketching Cylinders

1) Graph the curve in the trace $x = 0, y = 0$, or $z = 0$ depending on which variable is missing.
2) Draw that trace on your graph.
3) Draw a second trace (a copy of the curve in step 2) in a plane parallel to the trace.
4) Draw lines parallel to the x, y, or z axis passing through the two traces (this is determined by our missing variable).

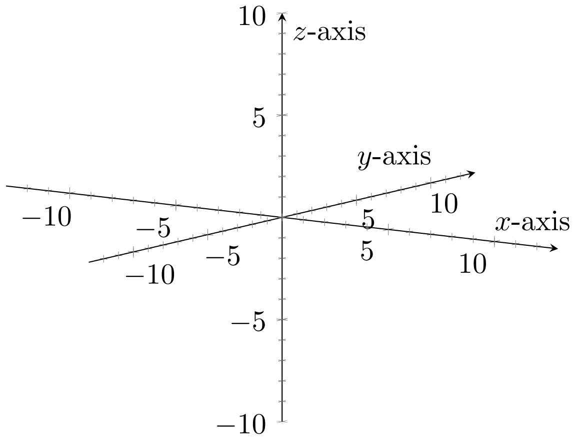
Example 9.8. *Sketch the curve* $x - \sin(z) = 0$*.*



Some Guidelines for Sketching:

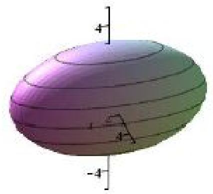
- Determine **Intercepts** (where the surfaces intersect coordinate axes).
- Determine any constraints on variables (for example: $x > 0$).
- Sketch the **Traces** or a good **cross section** (by setting one variable to 0 or a constant).
- Sketch at least two traces in parallel planes (for example traces with $z = 0$, $z = \pm 1$).
- Put it together in a Practice Sketch.
- Sketch it, sketch it real good!

Example 9.9. *Sketch the curve* $x = \dfrac{y^2}{4} + \dfrac{z^2}{9}$

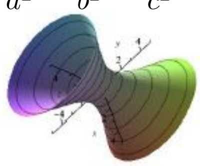


9.3 General Forms of Surfaces

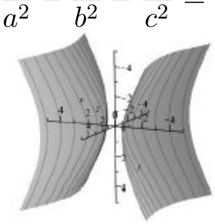
Ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



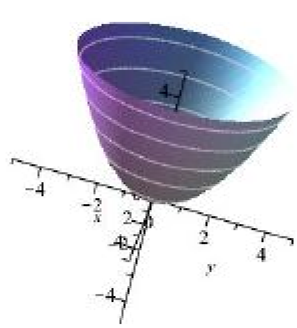
Hyperboloid of 1 sheet:
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$, $\dfrac{x^2}{a^2} + \dfrac{z^2}{b^2} - \dfrac{y^2}{c^2} = 1$
$\dfrac{y^2}{a^2} + \dfrac{z^2}{b^2} - \dfrac{x^2}{c^2} = 1$



Hyperboloid of 2 sheets:
$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$, $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} - \dfrac{z^2}{c^2} = 1$
$\dfrac{z^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{x^2}{c^2} = 1$



Elliptic Paraboloid:
$z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$, $y = \dfrac{x^2}{a^2} + \dfrac{z^2}{b^2}$, $x = \dfrac{y^2}{a^2} + \dfrac{z^2}{b^2}$

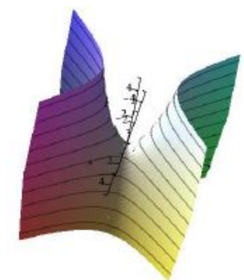


Hyperbolic Paraboloid:

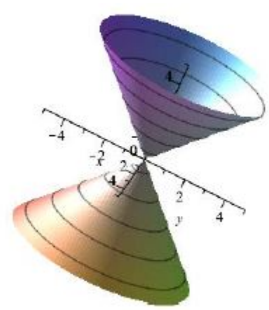
$z = \dfrac{y^2}{a^2} - \dfrac{x^2}{b^2}$ $\qquad z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

$y = \dfrac{z^2}{a^2} - \dfrac{x^2}{b^2}$ $\qquad y = \dfrac{x^2}{a^2} - \dfrac{z^2}{b^2}$

$x = \dfrac{y^2}{a^2} - \dfrac{z^2}{b^2}$ $\qquad x = \dfrac{z^2}{a^2} - \dfrac{y^2}{b^2}$



Elliptic Cone:
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$
$\dfrac{x^2}{a^2} + \dfrac{z^2}{b^2} - \dfrac{y^2}{c^2} = 0$
$\dfrac{y^2}{a^2} + \dfrac{z^2}{b^2} - \dfrac{x^2}{c^2} = 0$



9.4 ICE Surfaces

Match the graphs with the equations. Don't just look at your general forms, try to think about it using traces and cross sections. Then check with your general forms.

a) $y - z^2 = 0$

b) $2x + 3y - z = 5$

c) $4x^2 + \frac{y^2}{2} + z^2 = 9$

d) $x^2 + \frac{y^2}{2} - z^2 = 9$

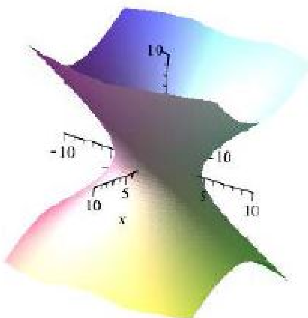
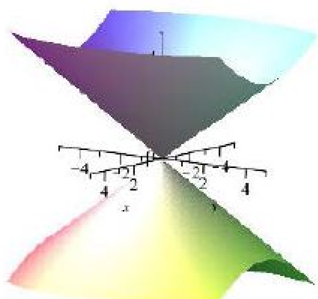
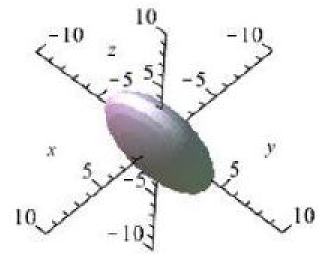
e) $x^2 + \frac{y^2}{2} = z^2$

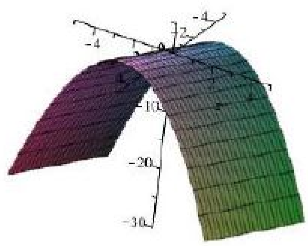
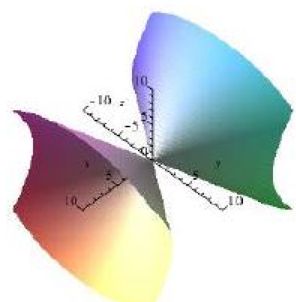
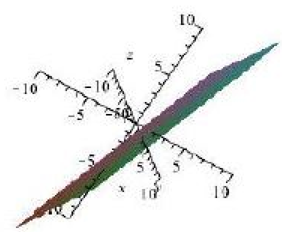
f) $x^2 + \frac{z^2}{2} - y^2 = 9$

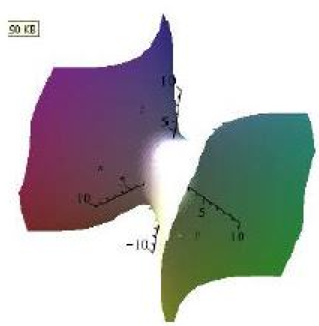
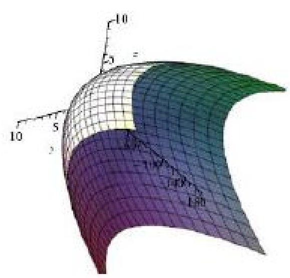
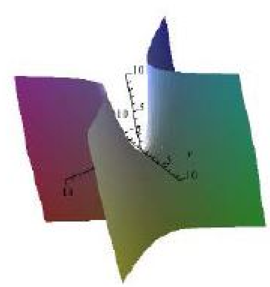
g) $z^2 + \frac{y^2}{2} = x^2$

h) $z = x^2 - y^2$

i) $9 = z^2 + y^2$

10 Multivariate Functions

10.1 Functions of Two Variables

So far, we have discussed real valued (scalar) functions and vector valued functions so far in this class. Today we introduce multivariate functions and in particular fuctions of two variables.

Notation: Functions of two variables are usually denoted *explicitly* as $z =$ ________ or ____________________ $= 0$.

Recall the *domain* of a function (unless specified) is the set of all input points (x, y) for which our function makes sense. The *range* of a function is the set of all real numbers z that are reached as our input points vary.

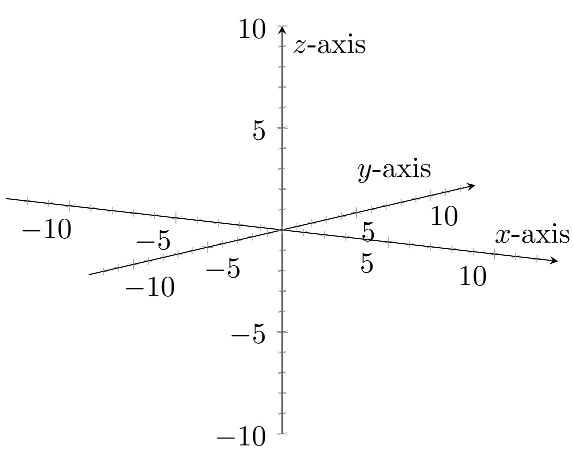
Example 10.1. *What is the domain of* $f(x, y) = \dfrac{\ln(y^2 - x)}{x - 2}$*?*

When we are asked to determine the *graph* of $f(x, y)$, we consider the surface ________. What is special about these surfaces is that we have a function, so each (x, y) corresponds with ________ ________ z. In other words, each line perpendicular to the xy plane intersects the surface at most ________ time.

Example 10.2. *Could we ever get an ellipsoid as the graph of a function of two variables?*

(a) Yes and I am very confident.
(b) Yes, but I am not very confident.
(c) No and I am very confident.
(d) No, but I am not very confident.

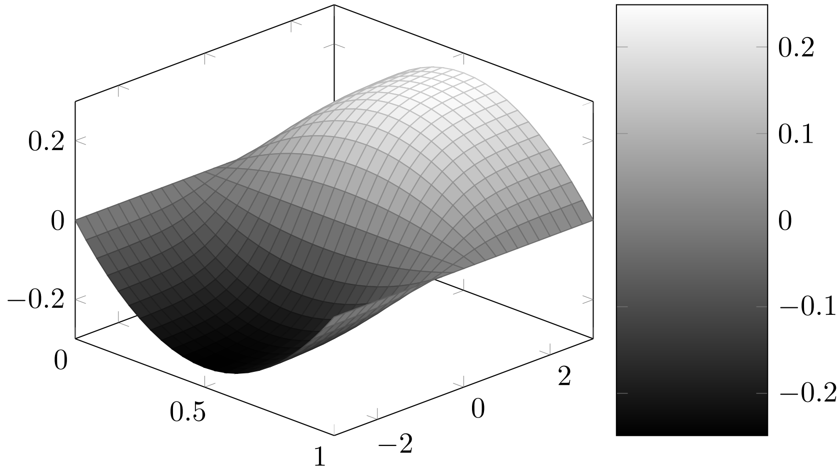
Example 10.3. *Sketch the graph of* $f(x, y) = \sqrt{4 + x^2 + y^2}$



Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

10.2 Level Curves and Contour Maps

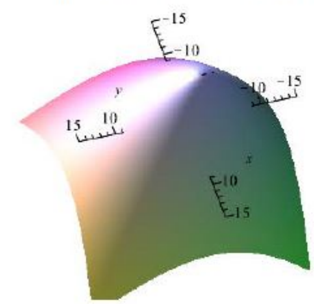
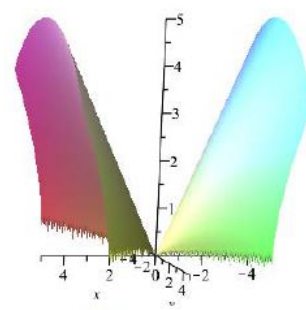
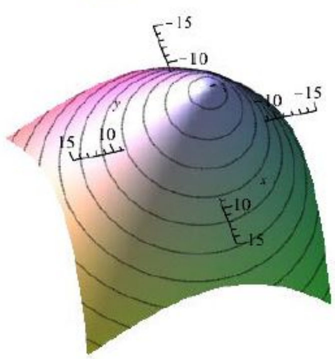
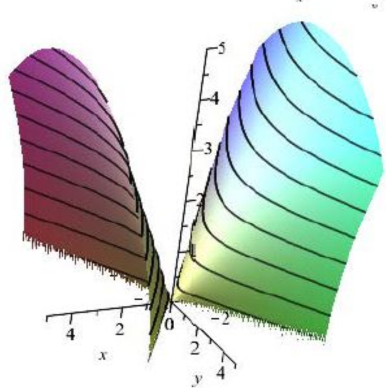
Sometimes sketching $f(x, y)$ in $\mathbb{R}^3$ makes people sad. Another way to represent or picture our surfaces are __________ maps.



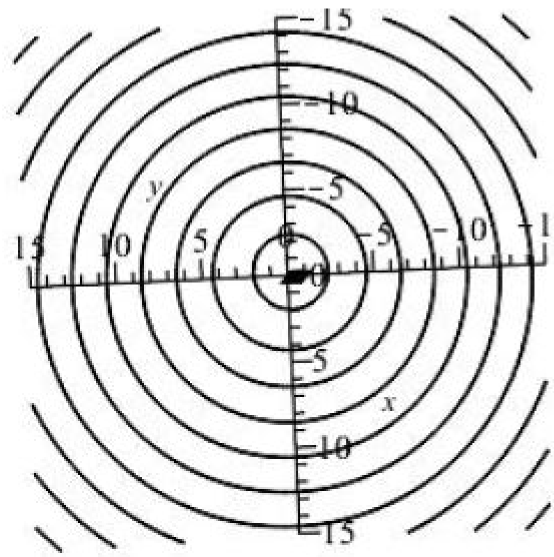
Definition: A level curve of the function $f(x, y)$ is the curve with the equation __________

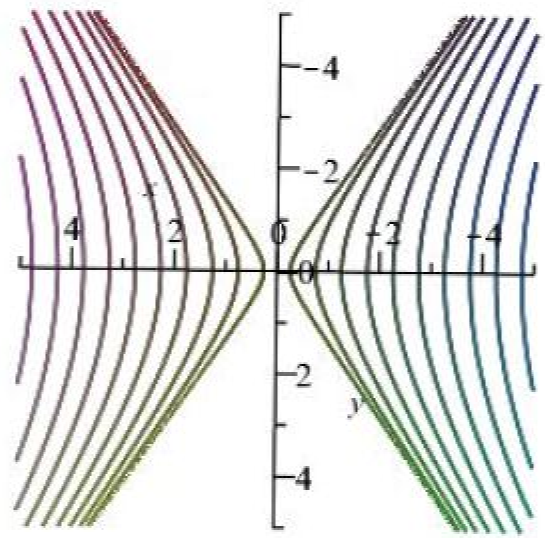
The set of level curves is called a __________ map.
Notice that when the level curves are closer together, the graph is __________.
And when the level curves are further apart, the graph is __________.

Examples: $f(x, y) = 9 - \sqrt{4 + x^2 + y^2}$ and $g(x, y) = \sqrt{x^2 - \frac{y^2}{2}}$

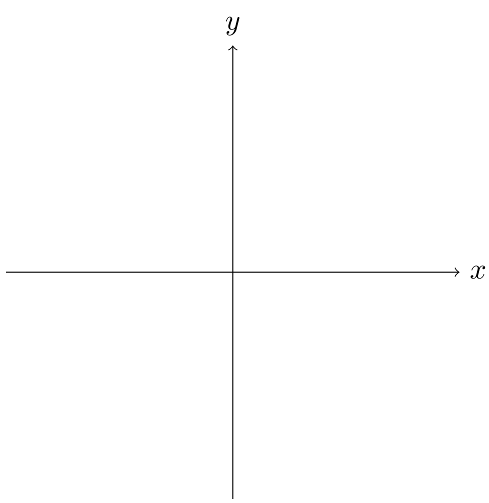





Contours for $f(x,y) = 9 - \sqrt{4 + x^2 + y^2}$ and $g(x,y) = \sqrt{x^2 - \dfrac{y^2}{2}}$.





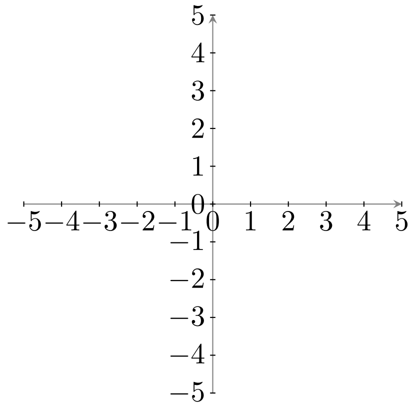
Example 10.4. *Draw the contour map for* $f(x,y) = x^2 + y^2$



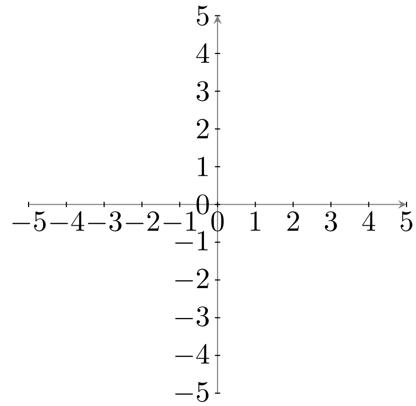
10.3 ICE Level Curves

For each of the following functions numbered 1-6, draw a contour map of the function by sketching the level curves for $k = 0, 1, 2, 3, 4$. Then match the contour maps with the associated graphs I-VI from the other side of the sheet.

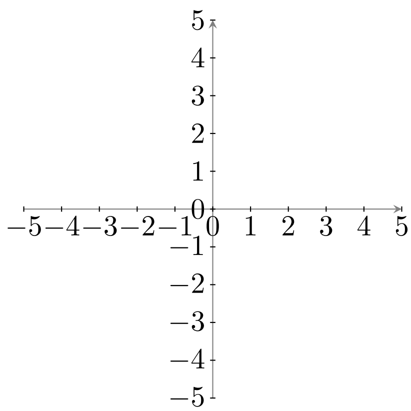
1. $f(x, y) = x - y$ Graph ___________



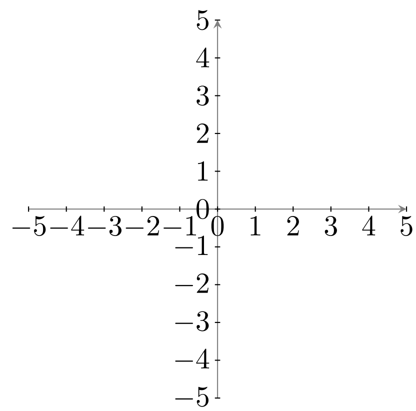
3. $f(x, y) = \sqrt{x^2 + y^2}$ Graph ___________



2. $f(x, y) = x^2 - y$ Graph ___________

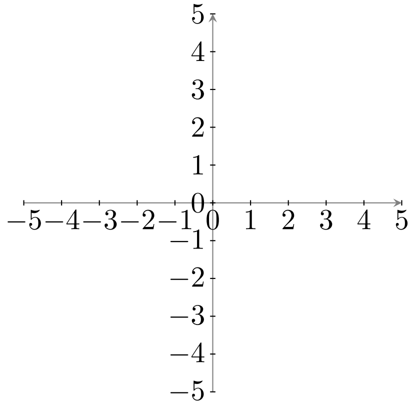


4. $f(x, y) = xy$ Graph ___________

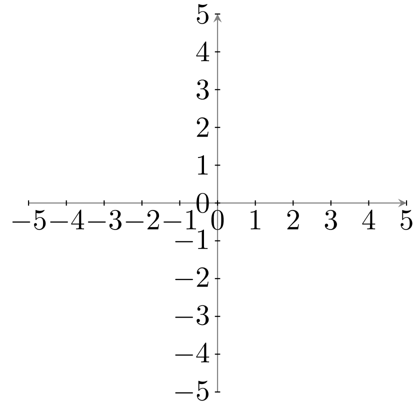


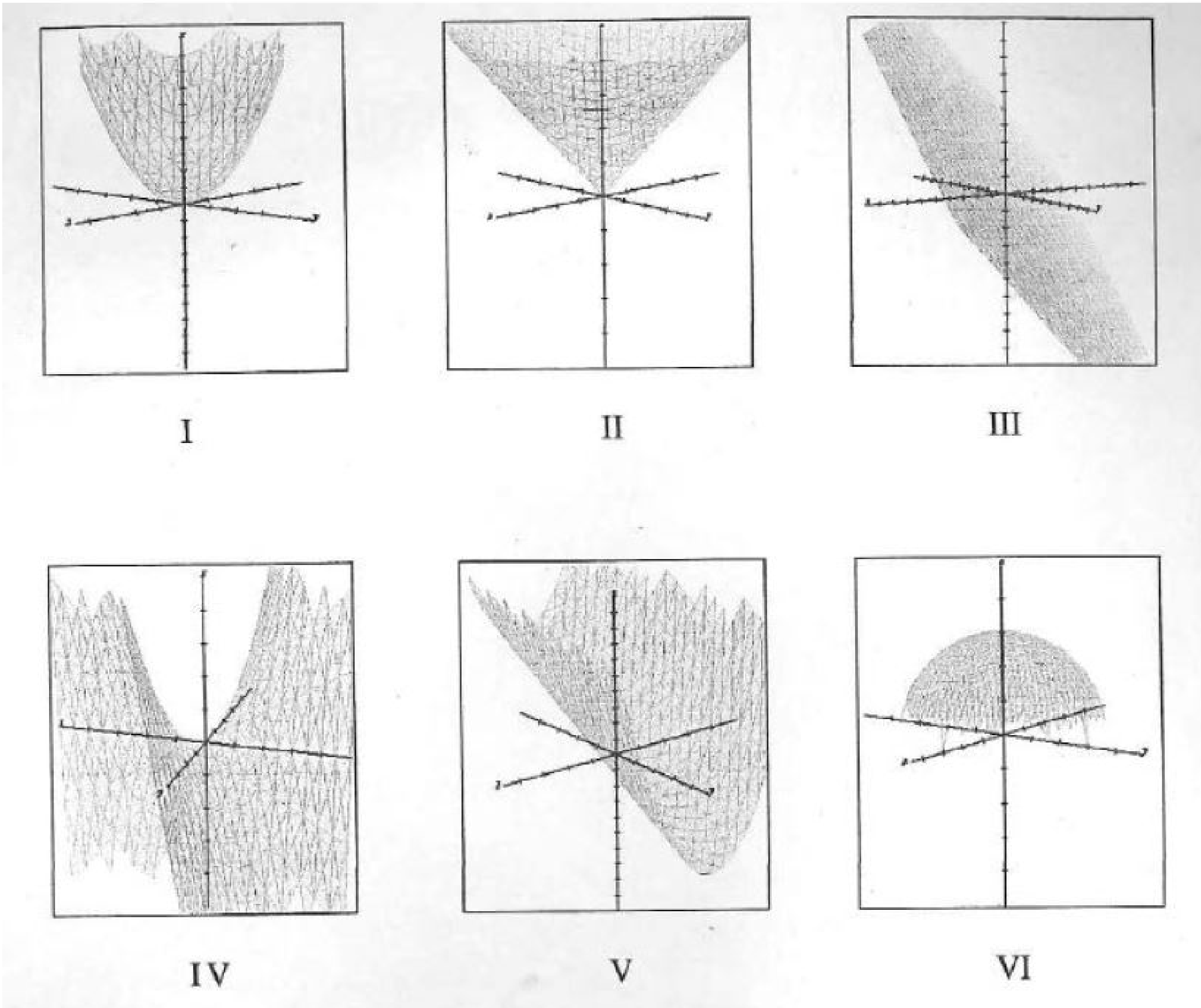
Turn page over for more.

5. $f(x,y) = x^2 + y^2$ Graph __________



6. $f(x,y) = \sqrt{9 - x^2 - y^2}$ Graph __________

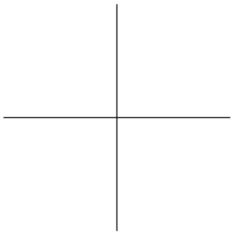



I
II
III
IV
V
VI

11 Polar Coordinates

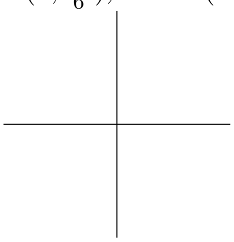
11.1 Defining Polar Coordinates

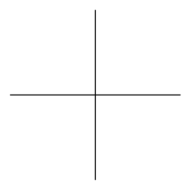
We are used to writing points in a two dimensional coordinate system using Cartesian coordinates: ________. Sometimes it is better to write a point using Polar Coordinates.



We pick a point called the ________ or origin and create the polar axis by drawing a horizontal ray from this point. Then we can define each point using an angle θ, and a radius r. Thus, we can write a point as ________.

Example 11.1. *Plot the points whose polar coordinates are the following:* $A(3, \frac{\pi}{6})$, $B(1, 5\pi)$, $C(2, \frac{7\pi}{6})$, *and* $D(-2, \frac{\pi}{6})$.





Converting Between Cartesian and Polar Coordinates:

Polar to Cartesian $(r, \theta) \to (x, y)$:
$x =$

$y =$

Cartesian to Polar $(x, y) \to (r, \theta)$:
$r^2 =$

$\tan\theta =$

Example 11.2. *a) Express the point with polar coordinates* $P(-2, \pi)$ *in Cartesian coordinates.*

b) Express the point with polar coordinates $P(3, \frac{3\pi}{4})$ *in Cartesian coordinates.*

Example 11.3. *Express the point with Cartesian coordinates $Q(3,3)$ in Polar coordinates.*

Example 11.4. *Which of the following are polar coordinates for the Cartesian point $Q(-1,\sqrt{3})$?*

a) $(2, \frac{5\pi}{6})$

b) $(-2, \frac{5\pi}{6})$

c) $(-2, \frac{-\pi}{3})$

d) $(-2, \frac{2\pi}{3})$

e) Math is the Bee's Knees!

Example 11.5. *Convert the following polar equations into Cartesian:*

a) $r^2 = 5$

b) $r\cos\theta = 5$

c) $r^2\sin(2\theta) = 1$

d) $r = \cos(\theta)$

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

11.2 ICE Polar Coordinates

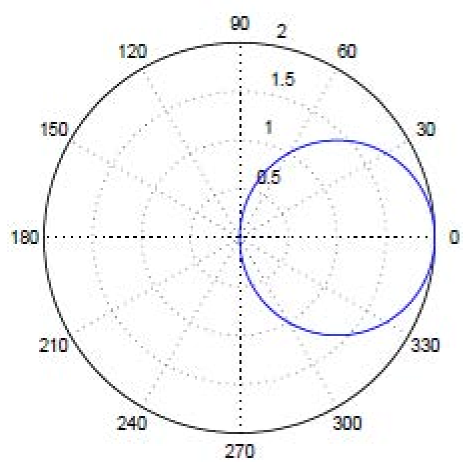
Voting Questions!

1. How do we represent the pole using polar coordinates?

2. True or False: A point in the xy-plane has a unique representation in polar coordinates.
   (a) True, and I am very confident.
   (b) True, but I am not very confident.
   (c) False, but I am not very confident.
   (d) False, and I am very confident.

3. The equation $\theta = \frac{\pi}{3}$ represents what type of graph in the Cartesian plane?
   (a) horizontal line
   (b) vertical line
   (c) a line through the origin
   (d) circle
   (e) I love Calculus 3!

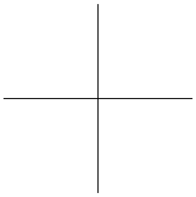
4. The graph below is which function expressed using polar coordinates?
   (a) $r = 2\sin\theta$
   (b) $r = 1$
   (c) $r = \cos\theta$
   (d) $r = 2\cos\theta$
   (e) $r^2 + \theta^2 = 1$

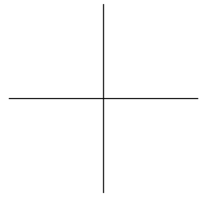


Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

11.3 Polar Curves

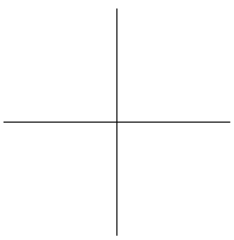
How could we represent a line through the pole using Polar coordinates?
How about a circle centered at the pole?

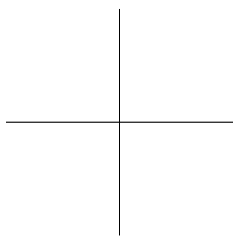


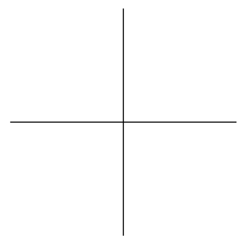


Example 11.6. *Describe the following polar curves.*

a) r=-4 b) $r = 4\sin\theta$. c) $r = \theta$







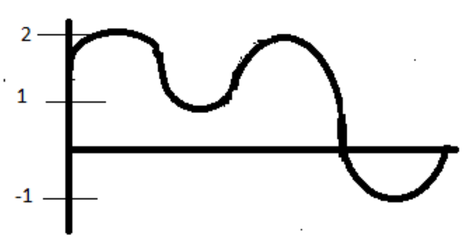
11.4 Graphing Polar Equations

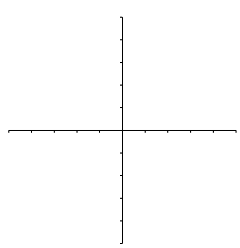
Plotting points is a time consuming way to graph polar equations. Often it is better to use the following:

Procedure: Cartesian to Polar Method for Graphing $r = f(\theta)$:

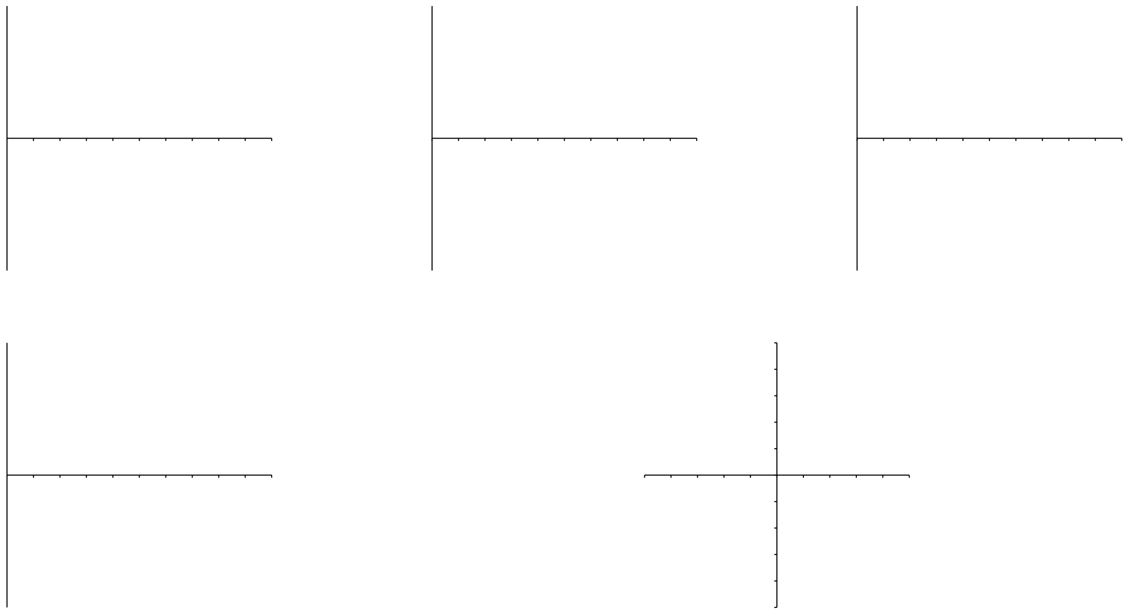
Step 1: Graph $r = f(\theta)$ as if r and θ were Cartesian coordinates with θ on the x-axis and r on the y-axis.
Step 2: Use the Cartesian graph in Step 1 as a guide to sketch the points (r, θ).

Example 11.7. *Use the graph of $r = f(\theta)$ on the left to sketch the polar equation.*

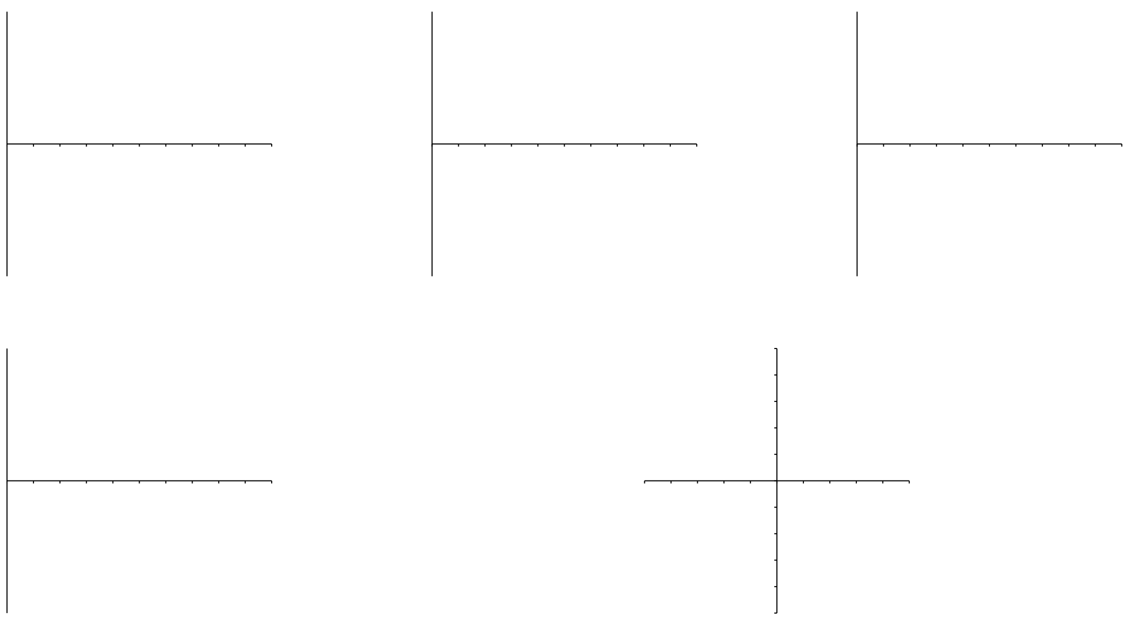




Example 11.8. *Graph* $r = 1 - \sin\theta$.



Example 11.9. *Graph* $r = 3\cos(2\theta)$.



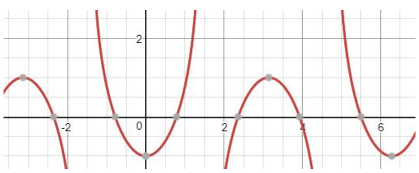
11.4.1 Circles In Polar Coordinates

The equation $r = a$ describes a circle of radius __________ centered at __________
The equation $r = 2a\sin\theta$ describes a circle of radius __________ centered at __________
The equation $r = 2a\cos\theta$ describes a circle of radius __________ centered at __________

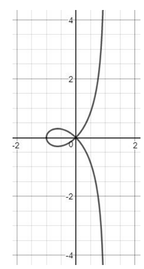
11.4.2 More Complex Curves In Polar Coordinates

There are more complicated polar equations you can graph. With the technology you can use today to graph these equations, it may seem like it is unnecessary to learn to graph polar curves by hand. I argue that it is good to understand the theory behind how we graph these equations because when in doubt you can always go back to the theoretical technique for complex examples.

For example suppose we wanted to graph $r = \sec\theta - 2\cos\theta$.
Below is the Cartesian graph $y = \sec x - 2\cos x$.

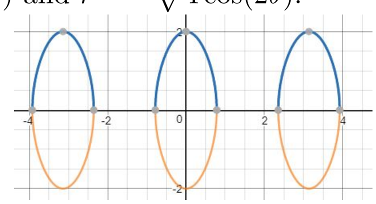


Thus the polar graph is:



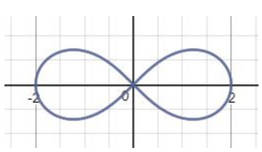
Sometimes we have to break up our equation into two parts. For example, when graphing $r^2 = 4\cos(2\theta)$, we should split it into two branches: $r = \sqrt{4\cos(2\theta)}$ and $r = -\sqrt{4\cos(2\theta)}$.

Here are the graphs of $y = \sqrt{4\cos(2x)}$ and $y = -\sqrt{4\cos(2x)}$:



Why do you suppose there are gaps in this graph?

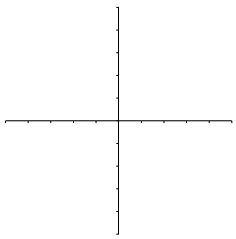
The polar graph for this “lemniscate” is



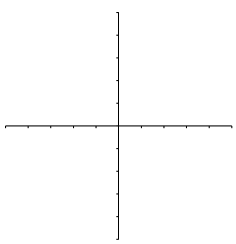
11.5 ICE Polar Graphs

1. Sketch the polar curves using the Cartesian to polar procedure to sketch the graph. You can check your sketch with your calculator or Desmos.

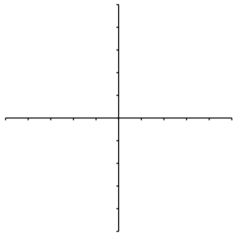
   a) $r = 1 + \cos(\theta)$. This is called a cardioid.



   b) $r = 1 + \cos(2\theta)$.



c) $r = 3\cos(\theta)$.



11.6 Derivatives and Tangents Lines in Polar Curves

When we find a tangent line for a polar curve $r = f(\theta)$, we think of θ as a __________. Then we can think of x and y in the following way:
$x = r\cos\theta \to x(\theta) =$
$y = r\sin\theta \to y(\theta) =$

Using the product rule we can find $x'(\theta) =$

Similarly $y'(\theta) =$

Then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} =$

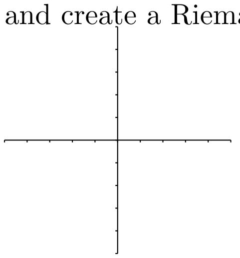
Horizontal tangents occur where $\dfrac{dy}{d\theta} =$__________, as long as $\dfrac{dx}{d\theta} \neq 0$.
Vertical tangents occur where $\dfrac{dx}{d\theta} =$__________, as long as $\dfrac{dy}{d\theta} \neq 0$.

How about second derivatives?

Example 4: Find the slope of the tangent line to $r = 2 - \sin\theta$ at $\theta = \frac{\pi}{3}$

11.7 Calculating Polar Areas

As one would expect, we can calculate the area of the region bounded by a polar equation. To derive this, we use the area of a sector of circle, $A = \frac{1}{2}r^2\theta$ and slice up up our region using sectors and create a Riemann sum.

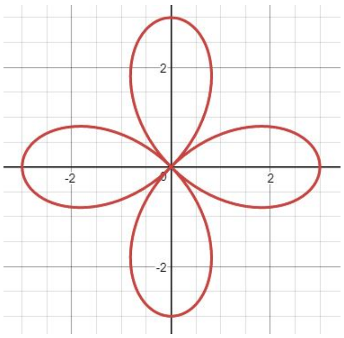


The Area of a polar region R bounded by $r = f(\theta)$ **between** $a \le \theta \le b$ is

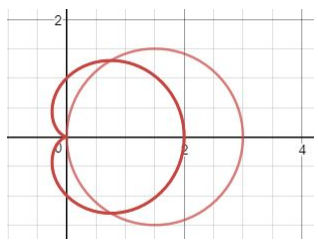
The Area of a polar region R bounded by $r = f(\theta)$ **and** $r = g(\theta)$ is

Note: The area is not always in form of $\int_1 - \int_2$. Can you draw an example of a region you would add to find the area between curves?

Example 11.10. *We graphed the polar curve $r = 3\cos(2\theta)$ in our notes last time. Set up an integral that represents the area enclosed by one of the loops. What can we use?*



Example 11.11. *In our ICE sheet, we graphed both $r_1 = 3\cos\theta$ and $r_2 = 1 + \cos\theta$.*



a) Find all points of intersection. Assume $0 \leq \theta \leq 2\pi$

b) Now set up an integral to find the area of the region that lies outside of $r_2 = 1 + \cos\theta$ and inside $r_1 = 3\cos\theta$. Recall they intersect at $\theta = \dfrac{\pi}{3}$.

Neat: Recall there is no elementary indefinite integral for the Gaussian function $f(x) = e^{-x^2}$. This function shows up in many applications including statistics and quantum mechanics. In Calculus II, you may have used numerical methods to approximate the Gaussian Function over finite intervals. You can actually use polar coordinates a little real analysis to show that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

11.8 Arc Length

Recall when we found the first derivatives of polar equations, we thought of θ as a parameter. Then using the product rule, we found that
$\frac{dx}{d\theta} = x'(\theta) =$ and $\frac{dy}{d\theta} = y'(\theta) =$

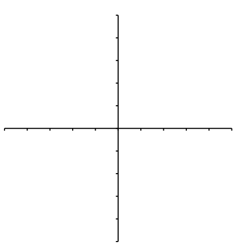
Now let's remember how we found the arc length for a parametric curve $x = f(t)$ and $y = g(t)$ for $\alpha \leq t \leq \beta$ where f' and g' are continuous and our curve is transversed once. We found our Arc Length=

So for $x = f(\theta)$, $y = g(\theta)$ for $a \leq \theta \leq b$,
we would have our Arc Length=

We can simplify $(\dfrac{dy}{d\theta})^2 + (\dfrac{dx}{d\theta})^2 =$ using $\cos^2\theta + \sin^2\theta = 1$, and get a more simplified version of the arc length for a polar curve.

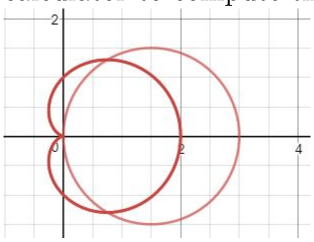
Thus the **Arc Length of a polar curve** $r = f(\theta)$ **for** $a \leq \theta \leq b$ is

Example 11.12. *Sketch the graph of and set up an integral to find the length of the curve* $r = e^{\theta}$ *for* $0 \leq \theta \leq 2\pi$.



11.9 ICE Polar Calculus

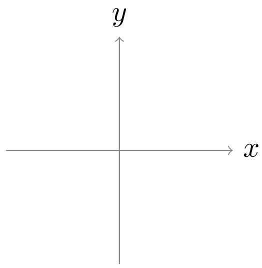
1. In our 11.11 Notes, we graphed both $r = 3\cos\theta$ and $r = 1 + \cos\theta$ and found all points of intersection of these curves. We also found the area inside the circle and outside $r = 1+\cos\theta$. Now set up an integral that find the area the region that lies in both curves. Use your calculator to compute this integral.

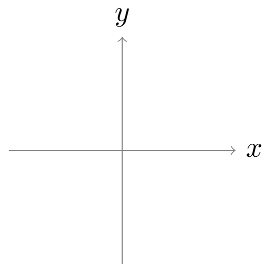


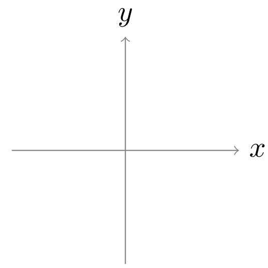
2. Find the points on $r = e^{\sqrt{3}\theta}$ where the tangent line is vertical.

12 Multivariate Limits and Continuity

The definition of a limit for multivariate functions is pretty much the same for single variable functions.
Definition: $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ if $|f(x,y)-L|$ can be made small if (x,y) is sufficiently close to (a,b). But now we have lots of paths...
Consider the paths to the origin...







Key Idea: We need to take our limit along all paths towards (a,b). If we can find two paths that both approach (a,b), but that go to different limits, then our limit ________ ________ ________!

Important: We can only follow paths that are in the ________________ of our function and that go to the limit point!

Remember: A limit can exist even if the function doesn't exist at a point.

12.1 Continuity of Multivariate Functions

Definition A function $f(x,y)$ is continuous at (a,b) if
f is defined at (a,b).

Theorem: If $f(x,y)$ is a polynomial or a rational function $\dfrac{p(x,y)}{q(x,y)}$ where $q(x,y)\neq 0$, then f is continuous.

Theorem: If $g(x,y)$ is continuous at (a,b) and f is continuous at $g(a,b)$ (as a single variable function), then $f\circ g$ is continuous at (a,b). So in other words, $\lim\limits_{(x,y)\to(a,b)} f(g(x,y)) =$

Key Idea: We need to be careful when we have points on the edge of domains.

12.2 Evaluating Multivariate Limits

Step 1: Check to see if the limit point is in the domain or on the boundary of the domain. If the former, we can ______________________________ otherwise, we need to try other methods.

Example 12.1. *Determine* $\lim\limits_{(x,y)\to(e^2,4)} \ln\sqrt{xy}$.

If the point is on the boundary of the domain, here are some techniques...

Key Technique 1: Sometimes you can separate the variables to turn the multivariable limit into a single variable limit which means we can use L'Hopital.

Example 12.2. *Determine* $\lim\limits_{(x,y)\to(0,3)} \dfrac{1-\cos(x)}{xy}$.

Example 12.3. *Archer, the math cat, trying to determine* $\lim_{(x,y)\to(a,b)} f(x,y)$. *He has found that this limit goes to 0 along the following paths:* $x = 0, y = 0, y = x$, *and* $y = x^2$. *He decides the limit must be 0. Is Archer's reasoning valid?*

a) Yes! This time Archer is right!
b) No!
c) I am not sure

Key Technique 2: To show a limit does not exist, try easy paths like along the lines __________, __________, __________, or __________. Note these paths must be in the domain and go to the limit point. For example, we can't use the path $y = 0$ if we are approaching $(0, 2)$.

Example 12.4. *Determine* $\lim_{(x,y)\to(0,0)} \dfrac{4xy}{2x^2 + y^2}$.

Key Technique 3: Sometimes limits at a point like $(0,0)$ may be easier to evaluate by converting to polar coordinates.
Remember that the same limit must be obtained as $r \to 0$ along all paths to $(0,0)$.

Example 12.5. *Determine* $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{5x^2+5y^2}$.

Example 12.6. *Determine* $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$.

Example 12.7. *What is the limit of* $\displaystyle\lim_{r\to 0} r\sin\theta$*?*

a) DNE
b) 0
c) 1
d) $\sin\theta$
e) To INFINITY and BEYOND!

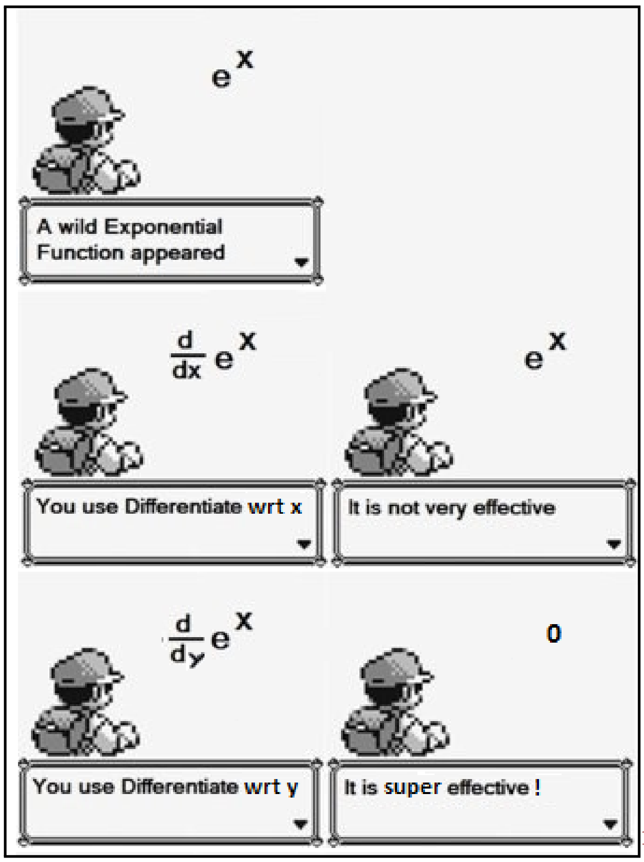
13 Partial Derivatives

Definition: Given $f(x, y)$, the *partial derivative with respect to* x at (a, b) is $f_x(a, b) =$

In general the x partial derivative is denoted by __________ or __________.

Key Idea: To find the partial derivative with respect to x, f_x, treat __________ as a constant and differentiate $f(x, y)$ with respect to __________.

Similarly we can find the partial derivative with respect to y by treating __________ as a constant and differentiate $f(x, y)$ with respect to __________.



[1]

Example 13.1. *Find f_x and f_y for $f(x, y) = xe^{2y}$*

[1]from https://www.reddit.com/r/funny/comments/1lxl7w/because_calculus/

Example 13.2. *Find* $f_x(-2,-1,8)$ *for* $f(x,y,z) = (\frac{xy}{z})^{\frac{1}{2}}$

13.1 Higher Partial Derivatives

$f_{xx} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2}$ = take the __________ partial derivative of __________

$f_{yy} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2}$ = take the __________ partial derivative of __________

$f_{xy} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x}$ = take the __________ partial derivative of __________

$f_{yx} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y}$ = take the __________ partial derivative of __________

Example 13.3. *Find all 4 second order partial derivatives for* $f(x,y) = xe^{2y}$

Notice $f_{xy} =$ __________ ! This is true all the time!*
*Well as long as f_{xy} exist and f_{yx} are continuous.

13.2 Thinking about Partial Derivatives

So far, we have described partial derivatives algebraically. Now let's look into how we come across them in other situations.

13.2.1 Partial Derivatives Numerically

Example 13.4. *The following table contains values for $f(x, y)$. Use it to estimate $f_x(3, 2)$ and $f_y(3, 2)$.*

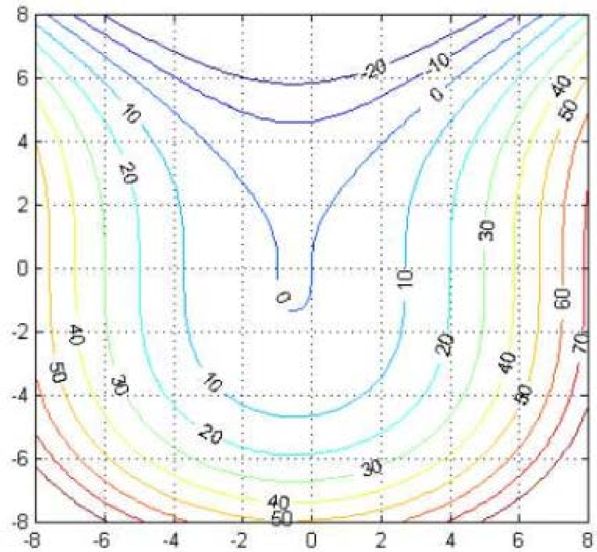
$x\backslash\backslash y$	1.8	2	2.2
2.5	12.5	10.2	9.3
3.0	18.1	17.5	15.9
3.5	20.0	22.4	26.1

13.2.2 Partial Derivatives Verbally

Example 13.5. *Let N be the number of applicants to a university, p the charge for food and housing at the university, and t the tuition. Suppose that N is a function of p and t such that $\dfrac{\partial N}{\partial p} < 0$ and $\dfrac{\partial N}{\partial t} < 0$. What information is gained by noticing that both partials are negative?*

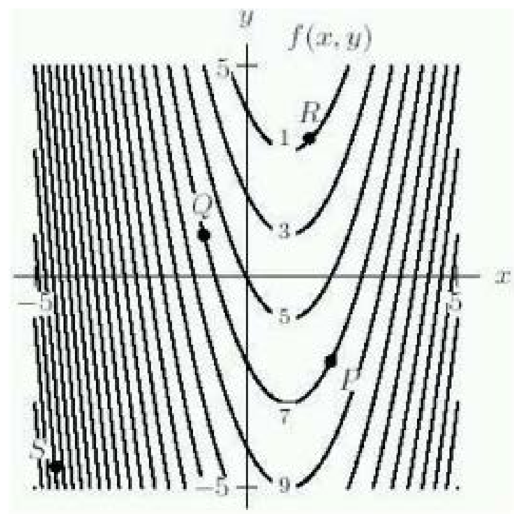
13.2.3 Partial Derivatives Graphically

Example 13.6. *Using the contour plot of $f(x, y)$, which of the following is true at the point $(4, 2)$?*



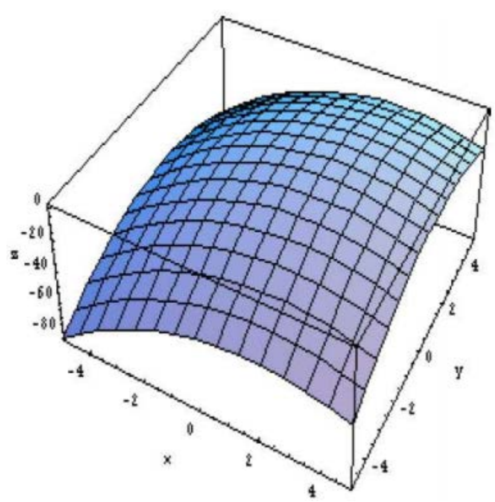
(a) $f_x > 0$ and $f_y > 0$
(b) $f_x > 0$ and $f_y < 0$
(c) $f_x < 0$ and $f_y > 0$
(d) $f_x < 0$ and $f_y < 0$

Example 13.7. *At point Q in the diagram below, which of the following is true?*



(a) $f_x > 0, f_y > 0$
(b) $f_x > 0, f_y < 0$
(c) $f_x < 0, f_y > 0$
(d) $f_x < 0, f_y < 0$

Example 13.8. *At which point above the xy plane to the right will both partial derivatives be positive?*



a) $(-5, 5)$
b) $(-5, -5)$
c) $(5, -5)$
d) $(5, 5)$
e) None are positive

Understanding Higher Partials Graphically :

Case 1: $f_{xx} > 0$ or $f_{yy} > 0$: Think "__________ __________"

- f_y or f_x is positive, and f is increasing at an increasing rate
- f_y or f_x is negative, and f is decreasing at a decreasing rate

Case 2: $f_{xx} < 0$ or $f_{yy} < 0$: Think "__________ __________"

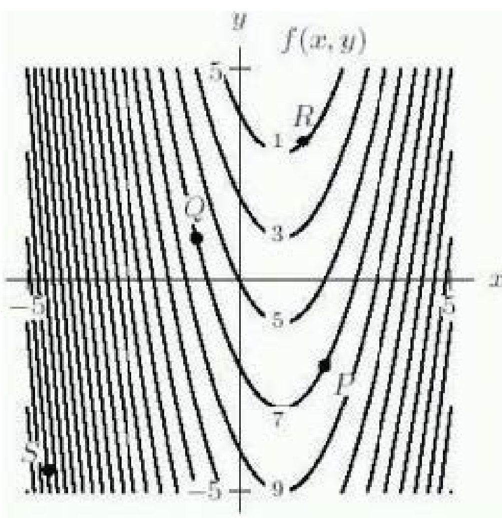
- f_y or f_x is positive, and f is increasing at a decreasing rate
- f_y or f_x is negative, and f is decreasing at an increasing rate

Case 3: $f_{xx} = 0$ or $f_{yy} = 0$: Think "no __________"

- f_y or f_x is positive, and f is increasing at a constant rate
- f_y or f_x is negative, and f is decreasing at a constant rate
- The point you are looking at is an "inflection point"

Case 4: Mixed Partials: To look at f_{xy} consider the rate of change of the slope in the x-direction as one moves in the y-direction.
For f_{yx} consider the rate of change of the slope in the y-direction as one moves in the x-direction.

Example 13.9. *At point Q in the diagram below, what are the signs of f_{xx}, f_{yy}, and f_{xy}?*



Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

13.3 Visualizing Partial Derivatives with Play-Doh

Please refer to Graphs Posted in Bb under Play Doh Activity.

Play Doh Assignment

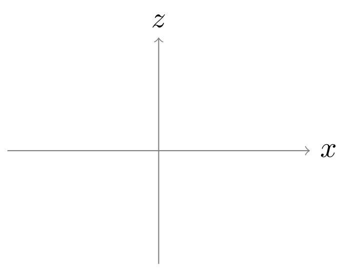
Directions: For each Contour plot, answer questions a-e.

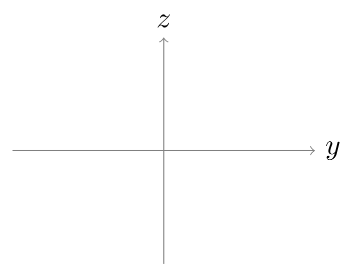
Contour Plot 1:

a) From plot, I think f_x is __________, f_y is __________, f_{xx} is __________, f_{yy} is __________, and f_{xy} is __________.

b) Represent the surface with Play Doh using the contour plot. Then pick which graph you think your contour plot/Play Doh surface represents. Attach picture of surface to your document that contains all pictures.

c) From my slice, Here are my my graphs on the xz and yz axes:





I think f_x is __________ and f_{xx} is __________. Attach picture of slice to your document that contains all pictures.

d) From my slice, I think f_y is __________ and f_{yy} is __________. Attach picture of slice to your document that contains all pictures.

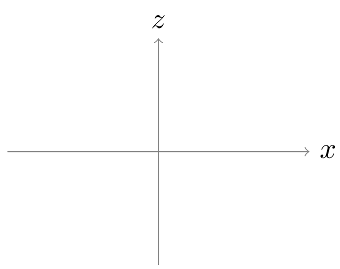
e) Pick the equation you think represents this function and determine the signs of your partial derivatives algebraically. I think my function is ____________________. From this, I found that f_x is __________, f_y is __________, f_{xx} is __________, f_{yy} is __________, and f_{xy} is __________.

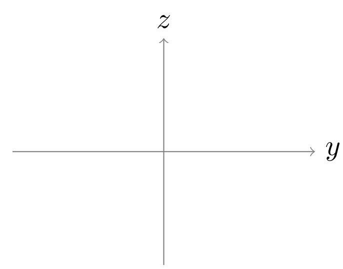
f) Did all of your choices for your partial derivatives agree? Why or why not, can you determine where you made a mistake if you made one? Did slicing help you determine the signs of your partial derivatives?

Contour Plot 2:
a) From plot, I think f_x is __________, f_y is __________, f_{xx} is __________, f_{yy} is __________, and f_{xy} is __________.

b) Represent the surface with Play Doh using the contour plot. Then pick which graph you think your contour plot/Play Doh surface represents. Attach picture of surface to your document that contains all pictures.

c) From my slice, Here are my my graphs on the xz and yz axes:





I think f_x is __________ and f_{xx} is __________. Attach picture of slice to your document that contains all pictures.

d) From my slice, I think f_y is __________ and f_{yy} is __________. Attach picture of slice to your document that contains all pictures.

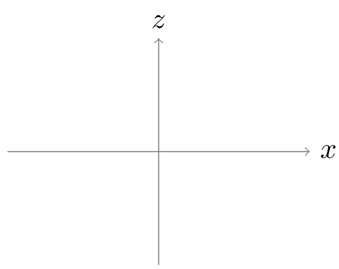
e) Pick the equation you think represents this function and determine the signs of your partial derivatives algebraically. I think my function is ____________________. From this, I found that f_x is __________, f_y is __________, f_{xx} is __________, f_{yy} is __________, and f_{xy} is __________.

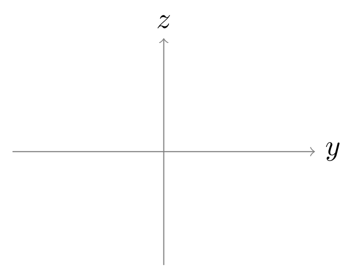
f) Did all of your choices for your partial derivatives agree? Why or why not, can you determine where you made a mistake if you made one? Did slicing help you determine the signs of your partial derivatives?

Contour Plot 3:
a) From plot, I think f_x is __________, f_y is __________, f_{xx} is __________, f_{yy} is __________, and f_{xy} is __________.

b) Represent the surface with Play Doh using the contour plot. Then pick which graph you think your contour plot/Play Doh surface represents. Attach picture of surface to your document that contains all pictures.

c) From my slice, Here are my my graphs on the xz and yz axes:





I think f_x is __________ and f_{xx} is __________. Attach picture of slice to your document that contains all pictures.

d) From my slice, I think f_y is __________ and f_{yy} is __________. Attach picture of slice to your document that contains all pictures.

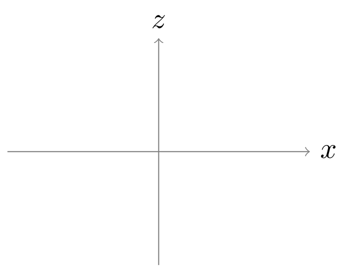
e) Pick the equation you think represents this function and determine the signs of your partial derivatives algebraically. I think my function is ____________________. From this, I found that f_x is __________, f_y is __________, f_{xx} is __________, f_{yy} is __________, and f_{xy} is __________.

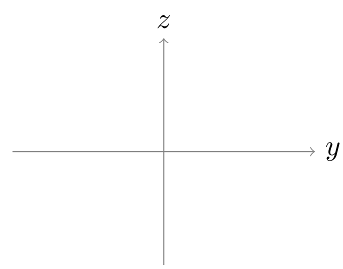
f) Did all of your choices for your partial derivatives agree? Why or why not, can you determine where you made a mistake if you made one? Did slicing help you determine the signs of your partial derivatives?

Contour Plot 4:
a) From plot, I think f_x is ________, f_y is ________, f_{xx} is ________, f_{yy} is ________, and f_{xy} is ________.

b) Represent the surface with Play Doh using the contour plot. Then pick which graph you think your contour plot/Play Doh surface represents. Attach picture of surface to your document that contains all pictures.

c) From my slice, Here are my my graphs on the xz and yz axes:





I think f_x is ________ and f_{xx} is ________. Attach picture of slice to your document that contains all pictures.

d) From my slice, I think f_y is ________ and f_{yy} is ________. Attach picture of slice to your document that contains all pictures.

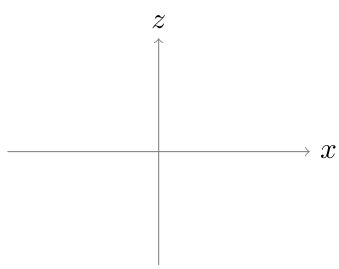
e) Pick the equation you think represents this function and determine the signs of your partial derivatives algebraically. I think my function is ________________. From this, I found that f_x is ________, f_y is ________, f_{xx} is ________, f_{yy} is ________, and f_{xy} is ________.

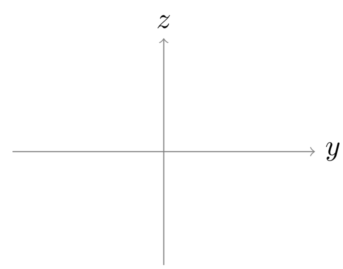
f) Did all of your choices for your partial derivatives agree? Why or why not, can you determine where you made a mistake if you made one? Did slicing help you determine the signs of your partial derivatives?

Contour Plot 5:
a) From plot, I think f_x is ________, f_y is ________, f_{xx} is ________, f_{yy} is ________, and f_{xy} is ________.

b) Represent the surface with Play Doh using the contour plot. Then pick which graph you think your contour plot/Play Doh surface represents. Attach picture of surface to your document that contains all pictures.

c) From my slice, Here are my my graphs on the xz and yz axes:





I think f_x is ________ and f_{xx} is ________. Attach picture of slice to your document that contains all pictures.

d) From my slice, I think f_y is ________ and f_{yy} is ________. Attach picture of slice to your document that contains all pictures.

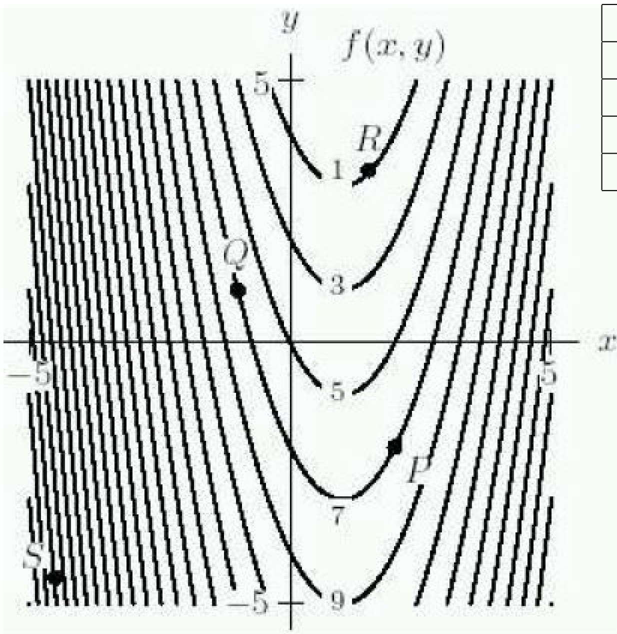
e) Pick the equation you think represents this function and determine the signs of your partial derivatives algebraically. I think my function is ________________. From this, I found that f_x is ________, f_y is ________, f_{xx} is ________, f_{yy} is ________, and f_{xy} is ________.

f) Did all of your choices for your partial derivatives agree? Why or why not, can you determine where you made a mistake if you made one? Did slicing help you determine the signs of your partial derivatives?

13.4 Ice Partial Derivative Review

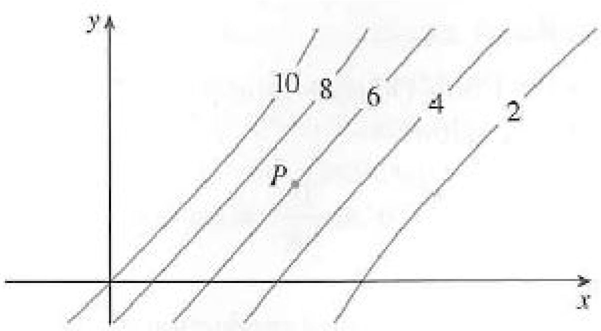
[2]

1. Determine the signs of f_x, f_y, f_{xx}, and f_{yy} at the points Q, R, and P from the picture below:



Q	**R**	**P**
f_x	f_x	f_x
f_y	f_y	f_y
f_{xx}	f_{xx}	f_{xx}
f_{yy}	f_{yy}	f_{yy}

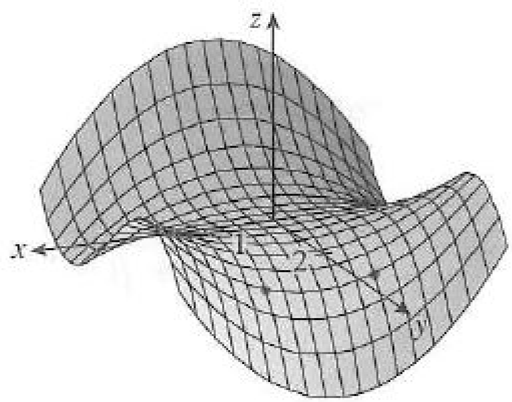
2. The level curves are shown for a function f. Determine whether the following partial derivatives are positive, negative, or zero at the point P.



f_x $\qquad$ f_y $\qquad$ f_{xx} $\qquad$ f_{yy}

[2]Questions 2 and 3 are taken from Stewart's *Calculus*

3. Determine the signs of the partial derivatives for the function at $(1,2)$ and $(-1,2)$:



$f_x(1,2)$ $f_x(-1,2)$

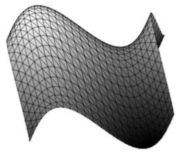
$f_y(1,2)$ $f_y(-1,2)$

$f_{xx}(1,2)$ $f_{xx}(-1,2)$

$f_{yy}(1,2)$ $f_{yy}(-1,2)$

14 Directional Derivatives and Gradient

We love partial derivatives f_x and f_y, but although they tell us a lot about our function, they don't tell us everything.Suppose I am standing on the surface below.



- What tells me the rate of change/slope of the surface in the direction parallel to the x axis?
- What tells me the rate of change/slope of the surface in the direction parallel to the y axis?
- What tells me the rate of change/slope of the surface in a direction other than a coordinate direction (like say northwest)?
- If I drop a ball at my feet on this surface and let it roll, in which direction will it roll?
- If I am hiking and I want to follow the steepest path, where do I step?

These latter questions will be answered in this section. (**Get excited!**)

Review: Recall, the direction of any vector $\mathbf{v}$ is determined by the unit vector $\mathbf{u} =$

Definition 1: Let $\mathbf{u} =< a, b >$ be a unit vector, then the *directional derivative of f at (x_0, y_0) in the direction of* $\mathbf{u}$ is $D_{\mathbf{u}}f(x_0, y_0) =$
Provided the limit exists!

Comment: If f is a differentiable function of x and y, then the Directional Derivative __________!

Why do we care? The directional derivative allows us the find the rate of change of f in an __________ direction!

Example 14.1. *a) Given $f(x, y) = e^{2xy}$, determine the directional derivative in the direction of* $\mathbf{i}$ *at (x_0, y_0).*

b) Given $f(x,y) = e^{2xy}$, determine the directional derivative in the direction of $\mathbf{j}$ *at* (x_0, y_0).

Our discovery in Example 14.1, allows us to define the directional derivative in terms of the __________.

It turns out we have a special name for $< f_x, f_y >$...

14.1 Gradient

Definition: Suppose f is differentiable at (a,b,c), the *gradient* of f at (a,b,c) is the vector valued function $\nabla f(a,b,c) =$

Recall we can denote $z = f(x,y)$ as $f(x,y,z) = 0$

Beware: $f'(x)$ is a __________, ∇f is a __________

Example 14.2. *Consider the surface* $f(x,y) = x^2y + y^2$.

a) Is f differentiable everywhere? Why or why not?

b) Find ∇f and $\nabla f(3,4)$.

Properties of ∇f:
1) $\nabla(f+g) =$
2) $\nabla(af) =$
31) $\nabla(fg) =$

14.2 Directional Derivatives Definition

Definition 2: Suppose f is a differentiable function of x & y, then the directional derivative in the direction of unit vector $\mathbf{u} =< a, b >$ is $D_{\mathbf{u}} f(x, y) =$
That is, $D_{\mathbf{u}} f(x, y) =$

Example 14.3. *Given $f(x, y) = e^y \sin(x)$, what is the directional derivative of f at $(\frac{\pi}{3}, 0)$ in the direction of $< \sqrt{3}, 1 >$?*

Example 14.4. *In which direction is the directional derivative of $z = x^2 + y^2$ at the point $(2, 3)$ most positive?*
(a) $\mathbf{i}$
(b) $\mathbf{i} - \mathbf{j}$
(c) $-\mathbf{i} + \mathbf{j}$
(d) $\mathbf{i} + \mathbf{j}$

Recall, $D_{\mathbf{u}} f(x, y) = \nabla f \cdot \mathbf{u} =$

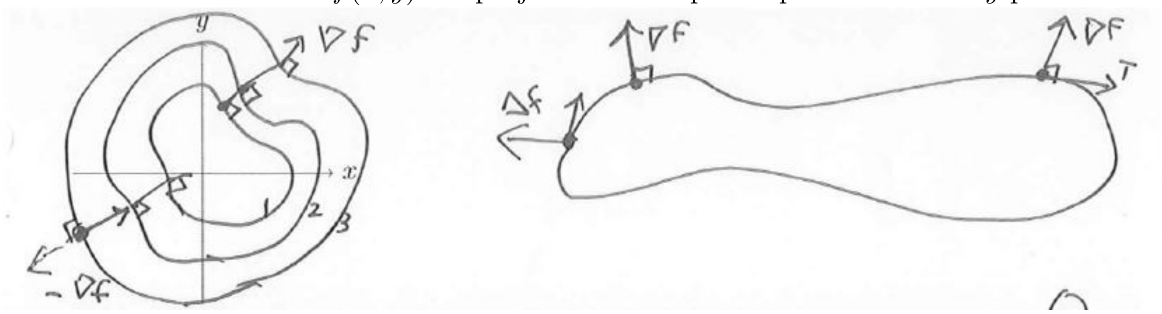
- At what angle will our directional derivative at (x_0, y_0) have its maximum value?
- When will the directional derivative have its minimum value?
- When will the direction derivative be 0?

Summary: Suppose f is a differentiable function at (x_0, y_0)

- f has its maximum rate of increase in the direction of __________ and this rate of increase is
- f has its maximum rate of decrease in the direction of __________ and this rate of decrease is
- The directional derivative is 0 in any direction that is __________ to ∇f

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Recall level curves of $z = f(x, y)$ are projections onto planes parallel to the xy plane.



Idea: Since the value of f is constant along each level curve, its rate of change is __________. Which means our directional derivative is __________. Thus $0 = D_{\mathbf{u}} f(x_0, y_0) =$
Thus the gradient is __________ to the level curves. In other words, it is __________ to the line tangent to the level curve.

Example 14.5. *Find a unit vector in the direction in which* $f(x, y) = e^{2x+y}$ *increases most rapidly at* $(0, 1)$.

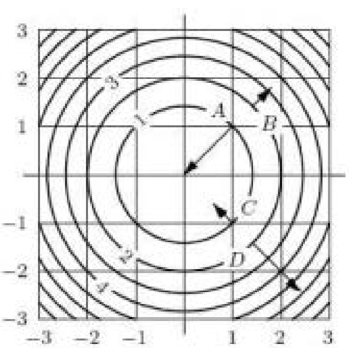
Example 14.6. *Let $f(x, y) = xy$.*
a)What is the maximum rate of change at $(0, 0)$.

b) Is the function identically zero near the origin? Can we talk about the direction of maximal change at $(0, 0)$?

Warning: If $\nabla f(x_0, y_0) =$ __________, then we can't determine the direction of maximal change at (x_0, y_0)

Example 14.7. *The surface of a hill is modeled by $z = 25 - 2x^2 - 4y^2$. When a hiker reaches the point $(1, 1, 19)$, it begins to rain. She decides to descend the hill by the most rapid way. Which of the following vectors points in the direction in which she starts her descent?*
(a) $< -4x, -8y >$
(b) $< 4x, 8y >$
(c) $< 4x, -8y >$
(d) $< -4x, 8y >$
(e) None of the above

Example 14.8. *Which of the vectors show on the contour diagram of $f(x, y)$ in the figure below could be ∇f at the point at which the tail is attached. (Pick one.)*
a) A *b) B* *c) C* *d) D* *I love Calc 3!*



Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

15 Linear Approximations of the Derivative

15.1 Tangent Plane Revisited

Recall, the general equation of a plane through point (x_0, y_0, z_0) with normal vector $< a, b, c >$ is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Definition: Consider $f(x, y, z) = 0$ where $f(x, y, z)$ is differentiable at (a, b, c) & $\nabla f(a, b, c) \neq \mathbf{0}$. Then *the equation of the tangent plane at the point* (a, b, c) is given by:

Definition: Consider $f(x, y) = z$ where f is differentiable at (a, b) & $\nabla f(a, b) \neq \mathbf{0}$. Then *the equation of the tangent plane at the point* (a, b) is given by:

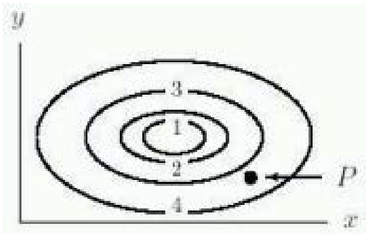
Example 15.1. *Find the equation of the tangent plane to the surface* $f(x, y) = x^2y + y^2$ *at* $(3, 4)$.

Example 15.2. *Suppose that* $f(x, y) = 2x^2y$. *What is the tangent plane to this function at* $x = 2$, $y = 3$*?*

(a) $z = 4xy(x - 2) + 2x^2(y - 3) + 24$
(b) $z = 4x(x - 2) + 2(y - 3) + 24$
(c) $z = 8(x - 2) + 2(y - 3) + 24$
(d) $z = 24(x - 2) + 8(y - 3) + 24$
(e) I want to go home. =)

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Example 15.3. *The figure below shows the contour map of the function $f(x, y)$. The tangent plane approximation to $f(x, y)$ at the point $P = (x_0, y_0)$ is*
$f(x, y) \approx c + m(x - x_0) + n(y - y_0)$. *What are the signs of $c, m,$ & n?*



(a) $c > 0, m > 0, n > 0$
(b) $c < 0, m > 0, n < 0$
(c) $c > 0, m < 0, n > 0$
(d) $c < 0, m < 0, n < 0$
(e) $c > 0, m > 0, n < 0$

15.2 The Differential

In Calculus I, we introduced the concept of the differential. Recall, the differential of $y = f(x)$ is $dy = f'(x)dx$. dy represents the amount that the __________ __________ rises or falls when x changes by amount $dx = \Delta x$. The change in y, Δy, represents the amount the __________ rises or falls when x changes by amount dx.

Let's draw the picture:

Why do we care? dy is a good approximation of __________ and can be much easier to find.

We now have the analogous situation in Calculus III, only now we have a tangent plane and not a tangent line.

Definition: Given a differentiable function $z = f(x, y)$, let dx and dy be the differentials of x and y respectively, then the *differential of f*, denoted as $df(x, y)$, df, or dz is

Sometimes this is called the *differential of z* or the differential of the *dependent* variable.

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Example 15.4. *Consider the surface $z = x^2 + y^2$.*
a) Use the tangent plane to the surface at $(3, 4)$ to estimate $f(2.9, 4.2)$.

b) What are Δz and dz in this case?

c) Use the tangent plane to the surface at $(3, 4)$ to estimate $f(2, 2)$.

d) What are Δz and dz in this case?

16 The Chain Rule

Chain Rule, Chain Rule, Chain Rule... we can't get enough of you! You are the best and most useful thing I know! -Anonymous Calculus Student

Chain Rule Version 1: Suppose z is a differentiable function of x & y. x and y are differentiable functions of t, then $\dfrac{dz}{dt} =$

Example 16.1. *a) Given* $z = \sqrt{x^2 + y^2}$ *where* $x = \cos(3t)$ *&* $y = \sin(3t)$*, determine* $\frac{dz}{dt}$.

b) Find $\frac{dz}{dt}$ *an alternative way.*

Chain Rule Version 2: Suppose z is a differentiable function of x and y and both x and y are function of s and t. (So in other words, we have $z =$ __________ and $x =$ __________ and $y =$ __________.) Then $z = f(x(s,t), y(s,t))$ has the following first partial derivatives:

$z_s =$ $z_t =$

Example 16.2. *Let $z = f(x,y) = e^{x+y}$ and let $x = st$ and $y = s - t$. What is the partial derivative of f with respect to s? With respect to t?*

Example 16.3. *Suppose $R = R(u,v,w)$, $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$. In the chain rule, how many terms will you have to add up to find the partial derivative of R with respect to x?*
(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

Example 16.4. *Suppose the radius of a right circular cone is decreasing at a rate of 2 in/s, while its height is increasing at a rate of 3 in/s. At what rate is the volume of the cone changing when the radius is 100 in. and the height is 120 in. (The volume formula for a right circular cone is* $V = \pi r^2 \dfrac{h}{3}$*.)*

Example 16.5. *A company sells regular widgets for $4 apiece and premium widgets for $6 apiece. If the demand for regular widgets is growing at a rate of 200 widgets per year, while the demand for premium widgets is dropping at the rate of 80 per year, the companys revenue from widget sales is:*

(a) staying constant
(b) increasing
(c) decreasing
(d) we cannot tell from this information
(e) I like math

16.1 Implicit Differentiation

In Calculus I, you learned how to differentiate a function of x and y. What was that process? Example: Find $\frac{dy}{dx}$ for $x^3 + 4xy^2 = y^3$

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Now consider if we think of $x^3 + 4xy^2 = y^3$ as $f(x, y) = 0$. In other words, we have $f(x, y) =$ differentiating both sides with respect to x using the chain rule like we did in calc I gives us $\dfrac{dy}{dx} =$

The Multivariate Way To Implicit Differentiation:

1) Set the equation/function equal to 0.
2) Find f_x.
3) Find f_y.
4) Then $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$.

Example 16.6. *Find* $\dfrac{dy}{dx}$ *for* $x^3 + 4xy^2 = y^3$ *using your Calc III method.*

Extension to 3 variables: Suppose $f(x, y, z) = 0$. Then using the chain rule and differentiating both sides with respect to x allows us to see that:

$\dfrac{\partial z}{\partial x} =$ ____________________ and $\dfrac{\partial z}{\partial y} =$ ____________________

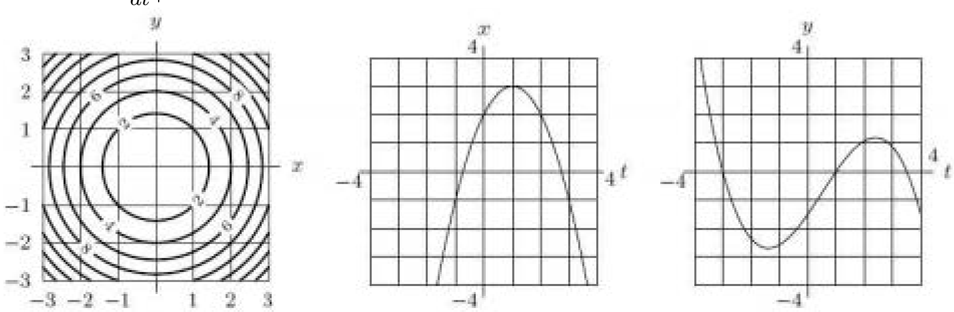
16.2 ICE Chain Rule

1. Consider $ye^{-x} + z\cos(x) = 0$

   a) Use the extension version of the chain rule to determine $\dfrac{\partial z}{\partial x}$.

   b)Determine $\dfrac{\partial x}{\partial z}$.

2. The figures below show contours of $z = z(x, y)$, x as a function of t, and y as a function of t. Decide if $\frac{dz}{dt}|_{t=2}$ is



(a) Positive
(b) Negative
(c) Approximately zero
(d) Cant tell without further information
(e) I like cats, could we have more of those?

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

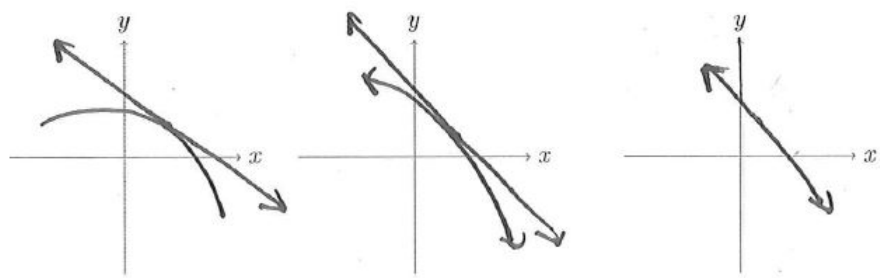
3. Use a tree diagram to write out the chain rule for the case where $w = f(t, u, v)$, $t = t(p, q, r, s)$, $u = u(p, q, r, s)$ and $v = v(p, q, r, s)$ are all differentiable functions.

17 Differentiability of Multivariate Functions

Even though we have been blissfully computing and approximating partial derivatives, we haven't talked about what it means for $f(x, y)$ to be differentiable.

Quick Review: If a function is continuous at $x = a$ what does that mean?

Also recall that derivatives are slopes of __________ lines.
What does our curve look like as we zoom in on the graph?



Recall, given $z = f(x, y)$, the **actual change in our function** at (a, b) is Δz=

Definition: $z = f(x, y)$ is *differentiable* at (a, b) if both __________ and __________ exist and the following exists:

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and ϵ_2 are functions that only depend on Δx (the change in x) and Δy (the change in y) and we have that $(\epsilon_1, \epsilon_2) \to (0, 0)$ as $(\Delta x, \Delta y) \to (0, 0)$. Of course a and b are fixed.

Definition: $f(x, y)$ is *differentiable on an open region R* if it is differentiable at __________ __________ in R.

Theorem: If $f_x(x, y)$ and $f_y(x, y)$ both exist and are both __________ on an open region R and (a, b) is a point in R, then $f(x, y)$ is ____________________ at (a, b).

Note: Just because $f_x(a, b)$ and $f_x(a, b)$ both exist, doesn't necessarily mean f is ____________________ at (a, b).

Theorem: If f is differentiable at (a, b), then f is __________ at (a, b).

Is the converse true? If f is continuous at (a, b), then f is differentiable at (a, b).

Contrapositive Thm: If f is NOT __________ at (a, b), then f is NOT __________ at (a, b).

Example 17.1. *Discuss the continuity and differentiability of*

$$f(x,y) = \begin{cases} \dfrac{4xy}{2x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Recall in our section on limits, we found that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{2x^2+y^2} =$ __________

18 Optimization

18.1 Max and Mins

Quick Review: How did we find max and minimum values in Calculus I?

Again, this section confirms that Calc III is in the same spirit of Calc I in its definitions of max and mins...

Definition: Suppose f is a continuous function on its domain D. Let p_0 be a point in D. Then we say $f(p_0)$ is a

- Global (Absolute) Maximum of f on D if $f(p_0) \geq f(p)$ for all p in the domain of f.
- Global (Absolute) Minimum of f on D if $f(p_0) \leq f(p)$ for all p in the domain of f.
- Local Maximum value if $f(p_0) \geq f(p)$ for all p in a nearby neighborhood of p_0.
- Local Mininimum value if $f(p_0) \leq f(p)$ for all p in a nearby neighborhood of p_0.

Note: If $f(p_0)$ is either a global max or min, we say it is a *global or absolute extreme value* and if $f(p_0)$ is either a local max or min, we say it is a *local extreme value*.

Recall from Calculus I: if we have a continuous function on a closed and bounded interval, what did our function have?

We have a similar result for Calculus III:

Theorem: If f is continuous on a __________ and __________ set S, then f attains both a __________ __________ and __________ __________ on S.

18.2 Critical Points

In Calc I, our max and mins were attained at __________ points or __________ points.

Definition: An interior point, (a, b) is a *critical point* if (a, b) is in the domain of $f(x, y)$ and either one of the following is true:

In Calc III, our Critical Points are of 3 types:

1. **Boundary Points:**
2. **Stationary Points:** an interior point at which f is differentiable and the $\nabla f =$ __________.
3. **Singular Points:** an interior point at which f is __________ differentiable at.

Theorem: Extreme values of $f(x, y)$ on a set S are also __________ points.

Process:

1. Find Critical Points
2. Test for Maximum and Minimums

Example 18.1. *Find the critical points of* $f(x, y) = x^4 + 2y^2 - 4xy$.

Example 18.2. *Find the critical points of* $f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - 4x - 9y$.

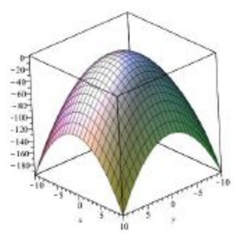
Example 18.3. *Which of these functions has a critical point at the origin?*
(a) $f(x,y) = x\cos y$
(b) $f(x,y) = x^2y + 4xy + 4y$
(c) $f(x,y) = x^2 + 2y^3$
(d) $f(x,y) = x^2y^3x^4 + 2y$
(e) All of the above

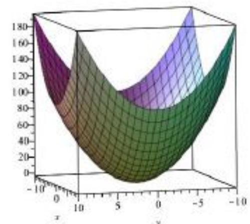
Second Derivative Test: Suppose the 2^{nd} partial derivatives of f are continuous for all points in an open disk at (a,b) where $f_x(a,b) = f_y(a,b) = 0$ and
let $D = D(a,b) =$

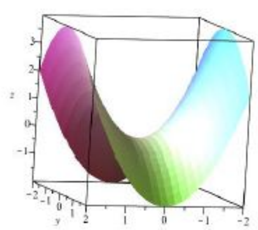
Then if

- $D(a,b) > 0$ AND $f_{xx}(a,b) < 0$, then (a,b) is a __________ __________
- $D(a,b) > 0$ AND $f_{xx}(a,b) > 0$, then (a,b) is a __________ __________
- $D(a,b) < 0$, then (a,b) is a __________ __________
- $D = 0$, then

Note: (a,b) is a saddle pt if we can find values such that $f(x,y) >$ __________ & $f(x,y) <$ _______.







Example 18.4. *Classify the critical points of* $f(x,y) = x^4 + 2y^2 - 4xy$. *Recall in Example 18.1 we found our CP's to be (0,0),(1,1) & (-1,-1) and* $f_x = 4x^3 - 4y$ *and* $f_y = 4y - 4x$.

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Example 18.5. *How would you classify the function* $f(x, y) = x^4 - y^4$ *at the origin?*

(a) This is a local maximum.
(b) This is a local minimum.
(c) This is a saddle point.
(d) We cannot tell.
(e) This is not a critical point.

Fact: If $f \geq 0$, f^n has the same __________ and __________ as f for any n.
Helpful hint: Sometimes it is easier to optimize f^n rather than f.

Example 18.6. *Find the point on the surface* $z^2 = x^2 + xy + y + 1$ *closest to the origin.*

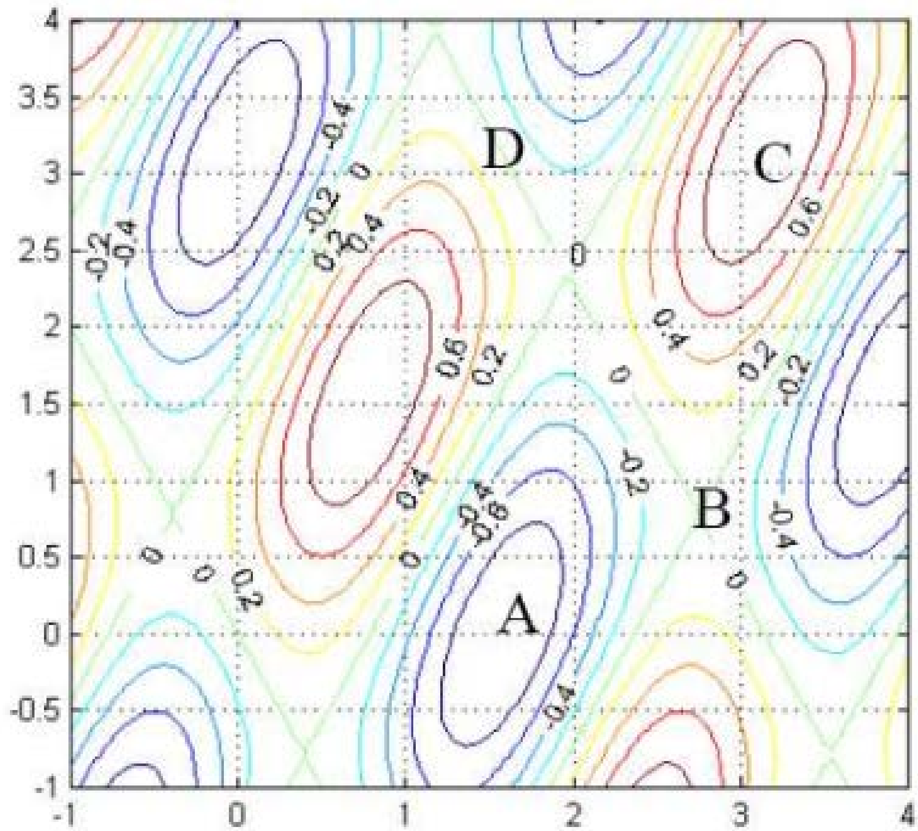
18.3 Using the Contour Map to Classify Critical Points

Classifying Critical Points using a Contour Map:

- Local Maxima: at the __________ of the contour and all contour __________ as we move towards the point.
- Local Minima: at the __________ of the contour and all contour __________ as we move towards the point.
- Saddle Point: Usually at the intersection of 2 contour lines. We increase in one direction and decrease when we move in "basically a perpendicular direction" (not necessarily exactly perpendicular). Idea: Go up one way and down in another direction.

Example 18.7. *Use the contour grid to determine which of the following points are critical points.*

(a) A and C
(b) A, C, and D
(c) A, B, and C
(d) A, B, C, and D
(e) None are critical Points



Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

18.4 ICE Optimization

Follow the steps below to find the absolute max and min values of $f(x, y) = x^2 + y^2 + x^2y + 4$ on the region $D = \{(x, y) : |x| \leq 1, |y| \leq 1\}$.

1. Sketch the region D.

2. Find the critical points of f, and then find the values of f at the critical points that are inside D.

3. On the top part of the boundary of D, f reduces to a function of one variable on a closed interval. Use techniques from Calculus I to find the extreme values of f on the top part of the boundary of D.

4. Repeat the previous question for the bottom part of the boundary of D.

5. Repeat the previous question for the right part of the boundary of D.

6. Repeat the previous question for the left part of the boundary of D.

7. The largest of the values from questions 2-6 is the absolute max and the smallest of these values is the absolute min. Find the absolute max and absolute min on D.

19 Lagrange Multipliers

19.1 Optimizing with Constraints

Goal: To find a max or min value of $f(x, y)$ subject to a constraint function $g(x, y) = c$.

We want to find the largest and smallest values of c such that the level curve $f(x, y) = k$ are tangent to __________.

When does this occur?

Theorem: Suppose $f(x, y)$ is a differentiable function in a region that contains the smooth curve given by $g(x, y) = 0$. If f has a local extreme value on the curve $g(x, y)$ at (a, b), then $\nabla f(a, b)$ is ____________________ to the line tangent to $g(x, y)$ at (a, b). Furthermore, as long as $\nabla g(a, b) \neq \mathbf{0}$, there exists a real number λ such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

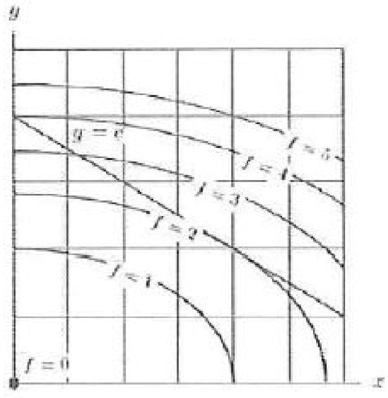
Method of Lagrange Multipliers:
Suppose f and g are differentiable on $\mathbb{R}^2$ with $\nabla g(x, y) \neq \mathbf{0}$ and $g(x, y) = 0$.
To find the max or min values of $f(x, y)$ subject to $g(x, y) = 0$:

Step 1: Find x, y, & λ such that __________ = __________ AND __________ = __________.

Step 2: From the different points (x, y) you found in Step 1, plug them into $f(x, y)$ and pick the largest value for your __________ and the smallest value for your __________.

Key Idea: $\nabla f = \lambda \nabla g$ at points in which the ∇f is parallel to ∇g which is when both ∇f and ∇g are orthogonal to the tangent line to the constraint function.

Example 19.1. *For $0 \leq x \leq 5$, find the maximum and minimum values of f on $g = c$.*



(a) max = 5, min = 0
(b) max = 4, min = 0
(c) max = 3, min = 2
(d) max = 4, min = 2

19.2 2 Constraints

Suppose we want to find extreme values of $f(x, y, z)$ and we have more than one constraint, say two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$. So we are looking for the max and min values of f which lie on the intersection of $g(x, y, z) = k$ and $h(x, y, z) = c$. In this case our ∇f is determined by both ∇g and ∇h. So we now have Lagrange multipliers for each constraint function. So in this example, we have

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

Therefore we now solve 5 equations:

Example 19.2. *Set up, but do not solve, the equations you would solve to find the extreme values of $f(x, y, z) = z$ subject to $x^2 + y^2 = z^2$ and $x + y + z - 24 = 0$*

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Example 19.3. *Find the maximum and minimum values of* $f(x, y) = \frac{1}{x} + \frac{1}{y}$ *subject to the constraint* $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 1$.

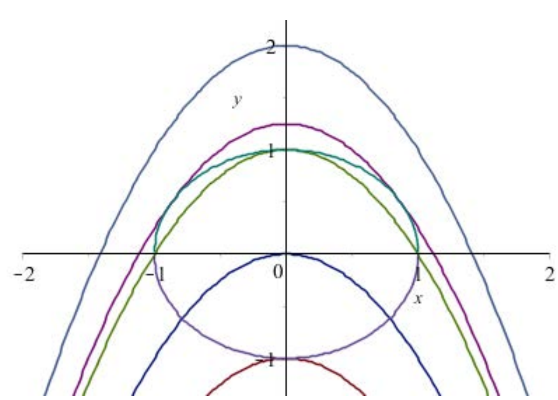
Example 19.4. *Find the minimum value of* $f(x, y) = x^2 + 4xy + y^2$ *subject to the constraint* $x - y - 6 = 0$.

Example 19.5. *Find the max and min values of* $f(x, y) = xy^2$ *subject to* $x^2 + y^2 = 1$.

19.3 ICE Lagrange Multipliers

1. Use Lagrange Multipliers to find the extreme values of the function $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$.

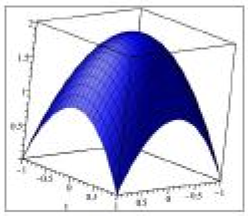
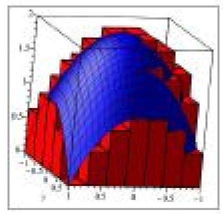
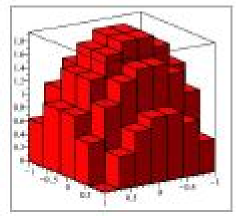
2. Let's explore this same question graphically: Below is the graph of the unit circle along with several level curves of the function f. Use the graph to determine the extreme values of the function $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$.



20 Multivariate Integration

20.1 Double Integrals over Rectangular Regions

When we were in Calculus I, we derived our definite integral by using a Riemann sum. In this case, we partitioned our interval $[a, b]$ and summed up the areas of rectangles. We have the same idea in Calculus III, only now instead of an integral, we have a rectangle, and we consider a small change in the area, ΔA. When evaluating the limit of a Riemann sum of $f(x, y)$ over a partitioned set rectangles, we find the volume of the solid bounded by $f(x, y)$ over the rectangular region.

Theorem: If f is bounded on a closed rectangle R and if it is continuous there except on a finite number of smooth curves, then f is __________ on R. In particular, if f is continuous on all of R, f is integrable.

Definition: If f is an integrable function, the *volume* of the solid bounded by $z = f(x, y)$ over the regional R is called the *double integral of f over R* and is denoted:

While evaluating double integrals over a rectangular region, we have similar results as with single integrals:

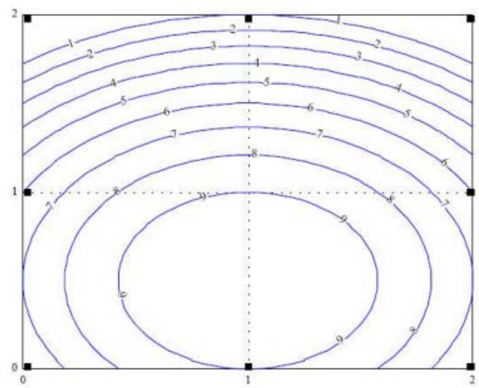
- It is linear: $\iint_R (k \cdot f(x, y) + g(x, y))dA =$
- Additive on non-overlapping rectangles: $\iint_R f(x, y)dA =$
- If $f(x, y) \leq g(x, y)$, then $\iint_R f(x, y)dA \leq$

Example 20.1. *Evaluate* $\iint_R (f(x, y)dA$ *for the function below*

$$f(x, y) = \begin{cases} 2, & 0 \leq x \leq 5, 0 \leq y \leq 1 \\ 3, & 0 \leq x \leq 5, 1 \leq y \leq 3 \end{cases}$$

Neat Fact: Notice that $\int_1^4 3dx = 3\cdot$(length of the interval)$= 3(4-1) = 9$. Similarly the integral of a constant function over a ________* Region can be calculated as $\iint_R CdA =$ ________
Note rectangular regions in this case mean integral with only finite values for the limits of integration. The region itself may not look rectangular.

Example 20.2. *Suppose the contour plot shown shows the height of a pile of dirt in feet. Which is a sum that is using the midpoint approximation rule to approximate the volume of the dirt pile?*



(a) $0\cdot 1 + 0\cdot 1 + 6\cdot 1 + 6\cdot 1$
(b) $9\cdot 1 + 9\cdot 1 + 9\cdot 1 + 9\cdot 1$
(c) $1\cdot 5 + 1\cdot 5 + 1\cdot 9 + 1\cdot 9$
(d) $6\cdot 1 + 6\cdot 1 + 9\cdot 1 + 9\cdot 1$

Notice in Example 20.1, we could have written our integral in the following way:

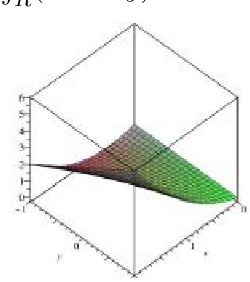
$$f(x,y) = \begin{cases} 2, & 0 \le x \le 5, 0 \le y \le 1 \\ 3, & 0 \le x \le 5, 1 \le y \le 3 \end{cases}$$

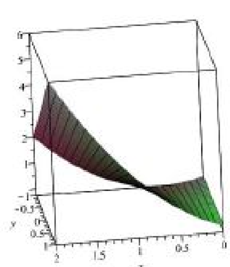
This form of the integral is called an ________ integral.

FUBINI's Theorem: Let $f(x,y)$ be continuous on a rectangular region $R = \{(x,y) : a \le x \le b, c \le y \le d\}$, Then $\iint_R f(x,y)dA =$

Warning: This is ONLY true for regions that are ________ !

Example 20.3. *Below we have the graph of the solid whose volume is given by the double integral over the rectangle $R = \{(x,y) : 0 \le x \le 2, -1 \le y \le 1\}$. Find the volume using an iterated integral. $\iint_R (x^2 + xy)dA$ and check that Fubini's Theorem holds.*



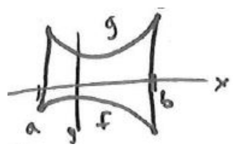


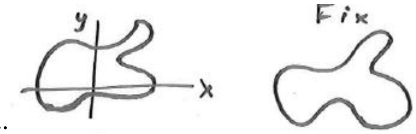
Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

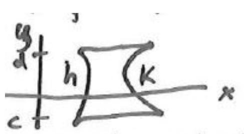
20.2 Iterated Integrals over General Regions

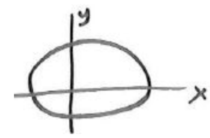
20.2.1 General (non-rectangular) Regions of Integration

Before we discussed integrating over rectangular regions which is nice because we can use __________!!! Unfortunately we often have more general regions

y-simple set: 

Neither: 

x-simple set: 

"Both": 

When in trouble, we can cut up a set so that we have a finite number of simple sets.

Key Idea: When setting up our iterated integral, make sure we have the __________ bounds on the outside.

20.2.2 Iterated Integrals

Theorem: Suppose R is bounded by 2 continuous functions of x (say $f(x) \leq y \leq g(x)$) and we have $a \leq x \leq b$, then $\iint_R f(x, y)dA =$

If R is bounded by 2 continuous functions of y (say $h(y) \leq x \leq l(y)$) and we have $c \leq y \leq d$, then $\iint_R f(x, y)dA =$

Note: If $f(x, y)$ is not a constant function, $\iint_R f(x, y)dA$ calculates the __________ under f over the region R. If $f(x, y) = 1$, $\iint_R f(x, y)dA$ calculates the __________ enclosed by R.

Calculating Area using double integrals: To calculate the **area** enclosed by $R := \{f(x) \leq y \leq g(x), a \leq x \leq b\}$, compute $\int_a^b \int_{f(x)}^{g(x)} dydx$

Example 20.4. *Set up an integral which calculates the area of the region bounded by* $R := \{\cos y \leq x \leq 5, 0 \leq y \leq 2\}$

Example 20.5. *Evaluate the volume of the solid made by* $f(x, y) = x + y$ *over the region in the* xy*-plane bounded by* $y = \sqrt{x}$ *and* $y = x^2$*.*

Separable Integrals *Sometimes* we can "convert" a double integral into a product of two single variable integrals. We call these **separable integrals**. We can do this when the limits of integration are **all** constants and we can write the integrand as a ("function of x")$\times$ ("function of y") $[f(x) \cdot g(y)]$. Below is an example of such an integral:

$$\int_0^5 \int_1^2 x^2 \cos(x) y^3 dy dx$$

Example 20.6. *which of the following integrals are separable:*

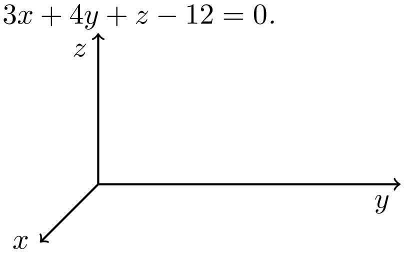
$$A = \int_0^5 \int_1^2 \cos(xy) dy dx, \quad B = \int_0^5 \int_1^{y^2} \cos(x) dy dx, \quad C = \int_0^5 \int_1^2 3 dy dx\,?$$

a) All of them
b) C
c) A and C
d) A and B
e) B and C

Example 20.7. *Change the order of integration of* $\int_0^2 \int_{x^2}^{2x} f(x,y)dydx$.

Example 20.8. *Why must we change the order of integration in order to evaluate* $\int_0^1 \int_y^1 e^{x^2} dxdy$*?*

Example 20.9. *a) Sketch the tetrahedron bounded by the coordinate planes and the the plane* $3x + 4y + z - 12 = 0$.



b) Set up an integral that represents the volume of the solid described in part a.

20.3 ICE Iterated Integrals

For each of the following iterated integrals, sketch the region of integration and rewrite the integral as an iterated integral with the order of integration reversed. DO NOT perform the integrations. **2 sides!**

1. $\displaystyle\int_0^1 \int_0^x xy^2 dydx$

2. $\displaystyle\int_0^1 \int_0^y xy^2 dxdy$

3. $\displaystyle\int_1^2 \int_0^{2x} xy^2 dydx$

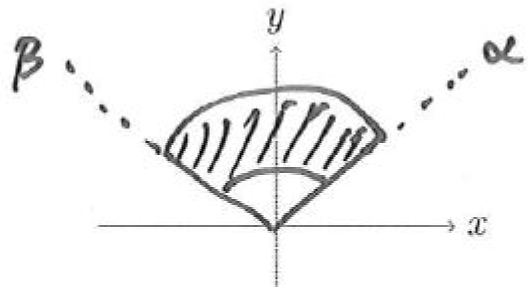
4. Write an integral for the volume under $f(x, y) = xy^2$ over the region of the unit circle in the $xy - plane$.

20.4 Integration over Polar Regions

Recall, we have the following conversions from Cartesian coordinates to and from polar coordinates:

$x =$ $\qquad\qquad$ $r^2 =$

$y =$ $\qquad\qquad$ $\theta =$



Theorem: Let f be continuous on the region given by $R = \{(r, \theta) : 0 \leq \alpha \leq \theta \leq \beta, f(\theta) \leq r \leq g(\theta)\}$ (note we need $\beta - \alpha \leq 2\pi$). Then $\iint_R f(r, \theta) dA =$

Area of Polar Regions: The area of a region given by $R = \{(r, \theta) : 0 \leq \alpha \leq \theta \leq \beta, f(\theta) \leq r \leq g(\theta)\}$ (note we need $\beta - \alpha \leq 2\pi$) can be calculated by $A = \displaystyle\iint_R DA = \int_\alpha^\beta \int_{f(\theta)}^{g(\theta)} r dr d\theta$

Note: This double integral gives us an __________ not a __________!

Example 20.10. *Evaluate* $\iint_R 4xy dA$ *for* $R = \{(r, \theta) : 1 \leq r \leq 3, 0 \leq \theta \leq \frac{\pi}{2}\}$.

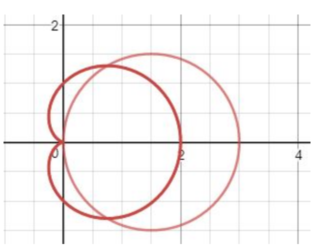
Example 20.11. *Sketch the region whose area is given by the integral* $\displaystyle\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_{1}^{3} r dr d\theta$.

Example 20.12. *Evaluate the integral* $\iint_R \sqrt{x^2+y^2} dA$ *where* $R = \{(x,y) : 1 \leq x^2+y^2 \leq 9, y \geq 0\}$ *by first changing to polar coordinates.*

Example 20.13. *a) Verify that the areas enclosed by* $r = 1$ *is* π *using a double integral.*

b) Verify that the area enclosed by the circle $2\cos(\theta)$ *is* π *using a double integral.*

Example 20.14. *In our 11.11 Notes, we graphed both* $r = 3\cos\theta$ *and* $r = 1 + \cos\theta$ *and found all points of intersection of these curves. We also found the area inside the circle and outside* $r = 1 + \cos\theta$*. Now set up a double integral to calculate the same value.*



21 Applications of Multivariate Integration

21.1 Density and Mass (Optional Topic)

Suppose we have a flat sheet that is soooo thin that we can treat it like it is a 2-dimenional object. Such a sheet is called a __________. You may remember this from Calc II or physics. We can use single integrals to compute the centroid or center of mass of a lamina. But we could only do this if the lamina had constant __________ or __________ that only depended on x. Now since we have double integrals, we can deal with densities that vary with respect to BOTH x and y.

Definition: Suppose that a lamina covers a region R in the xy-plane and has a density, $\delta(x, y)$, then the *total mass* of the lamina is $m =$

Example 21.1. *Find the mass of the lamina bounded by* $y = e^x$*,* $y = 0$*,* $x = 0$*, and* $x = 1$ *with density* $\delta(x, y) = x + y$*.*

When we have a constant density of K, we defined the moment of a particle about an axis as the product of its mass and its directed distance from the axis. Thus the moment or total moments about the y-axis: $M_y =$

the moment or total moments about the x-axis: $M_x =$

We used M_y and M_x to find the center of mass or __________ point.

Definition: Suppose that a lamina covers a region R in the xy-plane and has a density, $\delta(x, y)$, then the total moments about the y axis is $M_y =$

and the total moments about the x axis is $M_x =$

Thus the coordinates of the center of mass are $(\bar{x}, \bar{y})$,
where $\bar{x} =$ and $\bar{y} =$

The significance of the center of mass, is that the lamina behaves as if its entire mass is concentrated at its __________ __________ __________. AKA we can balance the lamina by supporting it at its center of mass.

What if we had a constant density? Then we should have something that coincides with the single variable version...

Example 21.2. *Find the total moment about the y axis of the homogeneous lamina with density k which covers the y-simple region: $R = \{(x, y)|a \leq x \leq b, g(x) \leq y \leq f(x)\}$.*

Example 21.3. *Find the center of mass of the lamina bounded by* $y = 0, y = \sqrt{4 - x^2}$ *with density* $\delta(x, y) = y$

21.2 Moments of Inertia/ Rotational Inertia (Optional Topic)

Definition: The *moment of inertia*(sometimes called the second moment) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the __________.

Idea: In physics the kinetic energy KE of our particle moving in a straight line with velocity v is:
$KE =$

If our particle is rotating about an axis with angular velocity w, its linear velocity is $v = rw$ where r is as defined above -the radius of the circular path. Thus $KE =$.

The r^2m portion of this is called the moment of inertia.

Definition: For a lamina with density $\delta(x, y)$ over region R, *the moment of inertia of the lamina about the x-axis* is
$I_x =$

and *the moment of inertia of the lamina about the y-axis* is
$I_y =$

and *the moment of inertia of the lamina about the z-axis* sometimes called the moment of inertia about the origin or the polar moment of inertia is
$I_z =$

Example 21.4. *Find the moments of inertia of the lamina about the x,y, and z axes for a homogeneous disk centered at the origin with radius a and with density $\delta(x, y) = k$.*

Remember we can find the mass of a homogeneous disk by $m =$ __________ $\times$ __________. Thus in our Example 4, we have that $Iz =$ __________

Definition: The *radius of gyration of a lamina about an axis* is the number R such that $mR^2 =$ __________ where __________ is the mass of the lamina and __________ is the moment of inertia about the given axis.

(So if $m\tilde{y}^2 = I_x$ and $m\tilde{x}^2 = I_y$, the point $(\tilde{x}, \tilde{y})$ is the point at which the mass of the lamina can be concentrated without changing the moments of inertia with respect to the coordinate axes.)

Example 21.5. *Find the radius of gyration about the x-axis from Example 21.4.*

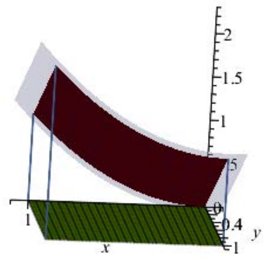
21.3 ICE Mass and Density (Optional Topic)

Find the center of mass of the lamina bounded by $r = 1, r = 2, 0 \leq \theta \leq \frac{\pi}{2}$ with density $\delta(r, \theta) = \frac{1}{r}$. Hint: use symmetry to find M_y.

21.4 Surface Area

In Calc II should have learned about finding the surface area of a revolution. We now develop a formula for the area of a surface defined by $z = f(x, y)$ over a region.

Idea: Partition our surface. These will be curved but we use the tangent plane to create a parallelogram. Then we find the areas of our parallelograms. Recall, the area of a parallelogram with adjacent sides **a** and **b** is



Visualizing surface area for $f(x,y) = x^2 + y$ on the rectangle $0 \le x \le 1, 0 \le y \le 1$. Included in the figure: the area (burgundy), the surface (navy), the domain (leafgreen).

Definition: Given a smooth *parametric surface* S given by the equation $\mathbf{r}(u, v) =< x(u, v), y(u, v), z(u, v) >$ (where S is covered just one as the parameters range through the domain of S), then the surface area of S is given by $A(S) = \iint_D |r_u \times r_v| dA$ Note $r_u =< x_u, y_u, z_u >$ and $r_v =< x_v, y_v, z_v >$.

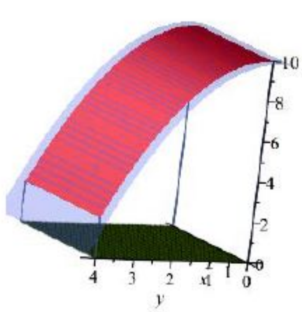
Explicit Case: To find the Area of a surface represented explicitly by $z = g(x, y)$, we evaluate

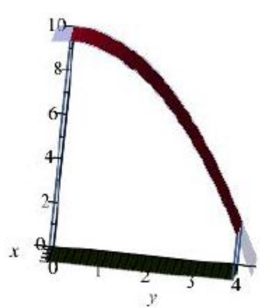
$$A(S) = \iint_R \sqrt{1 + g_x^2 + g_y^2} dA$$

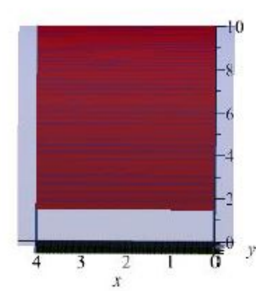
Note: Often setting up a surface area problem is easy, but often it is difficult or impossible to evaluate these integrals. So we must use a computing system or approximate the integral.

Example 21.6. *a) Use the following 3 views of the surface to determine a good approximation of the portion of the graph over our square. Find the surface area of the portion of the graph of* $f(x, y) = 10 - \dfrac{y^2}{2}$ *lying over the square with vertices* $(0, 0), (4, 0), (4, 4),$ *&* $(0, 4)$.

16 ***200*** ***-100*** ***72*** ***36***







b) Find the surface area of the portion of the graph of $f(x,y) = 10 - \dfrac{y^2}{2}$ *lying over the square with vertices* $(0,0), (4,0), (4,4),$ *&* $(0,4)$.

Example 21.7. *Find the surface area of the part of the paraboloid* $x = y^2 + z^2$ *that lies inside the cylinder* $y^2 + z^2 = 9$.

21.5 ICE Surface Area

1. Set up an integral to find the surface area of the part of $z = 15 - x^2 - y^2$ above the plane $z = 14$

2. Set up an integral to find the part of $z = 9 - x^2$ above the xy plane with $0 \leq y \leq 20$.

22 Triple Integrals

"Triple Integrals will make you smile. Triple Integrals, they last a while. Triple Integrals will help you, mister to punch tough volumes right in the kisser. Triple Integrals!"

22.1 Triple Integrals in Cartesian Coordinates

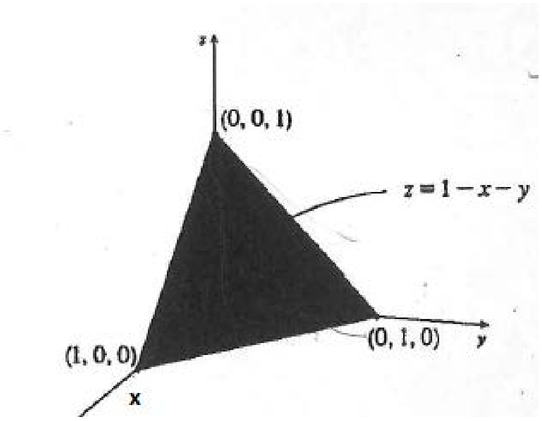
Triple integrals are "basically" double integrals with an extra variable.
Whereas $\iint_R 1dA =$ __________

We now have $\iiint_D 1dV =$ __________

Now Let $f(x, y, z)$ be a continuous function defined on a region R where $R = \{(x, y, z) : a \leq x \leq b, g(x) \leq y \leq h(x), G(x, y) \leq z \leq H(x, y)\}$ where $g, h, G,$& H are continuous. Then the triple integral of f on R is evaluated as the following iterated integral:

Example 22.1. *Find the volume of the snow cone made where the top of the cone is* $x^2+y^2+z^2 = 8$ *and the cone part is given by* $z = \sqrt{x^2 + y^2}$.

Example 22.2. *Set up a triple integral for the volume of the following solid. (The tetrahedron created by the intersection of the plane $z = 1 - x - y$ and the coordinate planes.)*

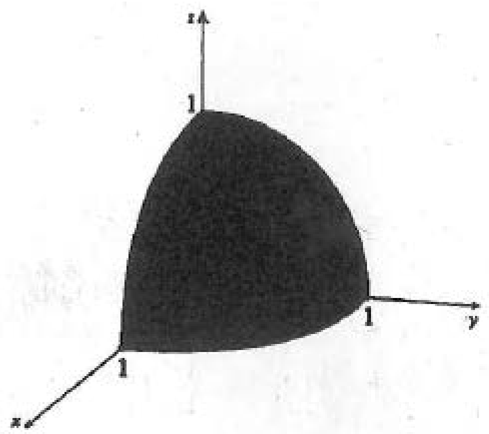


Example 22.3. *Sketch the solid whose volume is given by the iterated integral* $\int_0^1 \int_0^{1-x} \int_0^{2-2z} dydzdx$.

Example 22.4. *Consider the integral* $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)dzdydx$. *Rewrite this integral as an integral of the form:* $\int\int\int f(x,y,z)dydzdx$.

22.2 ICE Triple Integrals

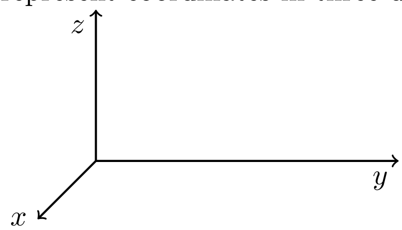
1. Set up a triple integral for the volume of the following solid, which is the piece of the unit sphere in the first octant.



22.3 Triple Integrals Using Cylindrical and Spherical Coordinates

22.3.1 Cylindrical Coordinates

Recall, when we are dealing with a 2-dimensional coordinate system, we could write points in Cartesian or Polar form. Today we will discuss two alternate coordinate systems for which we can represent coordinates in three dimensions.



So basically we have Polar Coordinates with a ________!

Converting Between Cartesian and Cylindrical Coordinates:

Cartesian to Cylindrical: $(x, y, z) \to (r, \theta, z)$:

$r =$

$\theta =$

$z =$

Cylindrical to Cartesian: $(r, \theta, z) \to (x, y, z)$:

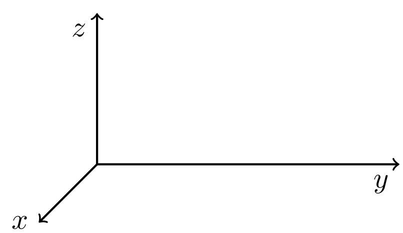
$x =$

$y =$

$z =$

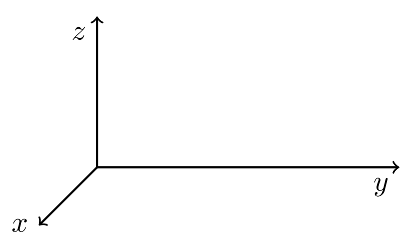
Example 22.5. *What are the Cartesian coordinates of the point with cylindrical coordinates* $(r, \theta, z) = (4, \pi, 6)$*?*

(a) (x, y, z) = (0, -4, 4)
(b) (x, y, z) = (0, 4, 6)
(c) (x, y, z) = (-4, 4, 4)
(d) (x, y, z) = (4, 0, 4)
(e) (x, y, z) = (-4, 0, 6)



Example 22.6. *What are the cylindrical coordinates of the point with Cartesian coordinates* $(x, y, z) = (3, 3, 7)$*?*

(a) $(r, \theta, z) = (3, \pi, 7)$
(b) $(r, \theta, z)) = (3, \frac{\pi}{4}, 3)$
(c) $(r, \theta, z) = (3\sqrt{2}, \frac{\pi}{4}, 7)$
(d) $(r, \theta, z) = (3\sqrt{2}, \pi, 7)$
(e) $(r, \theta, z) = (3\sqrt{2}, \pi, 3)$

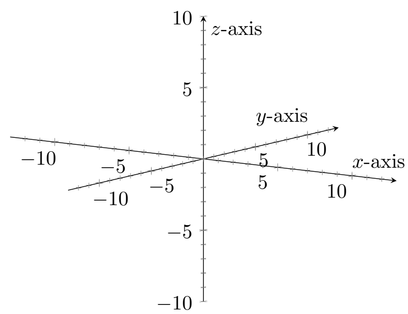


Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Example 22.7. *Write the equation of the surface $z = x^2 + y^2 - y$ in cylindrical coordinates.*

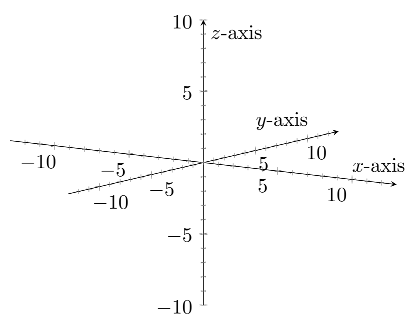
Example 22.8. *The graph of $r = 5$ is a*

(a) circle
(b) sphere
(c) cone
(d) plane
(e) cylinder

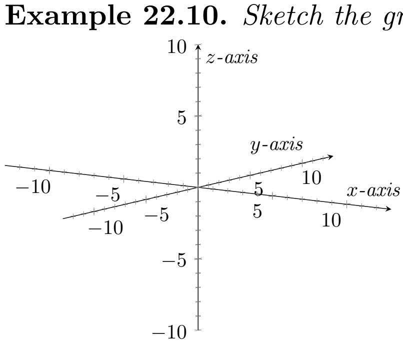


Example 22.9. *The graph of $\theta = \frac{\pi}{4}$ is a*

(a) circle
(b) sphere
(c) cone
(d) plane
(e) cylinder



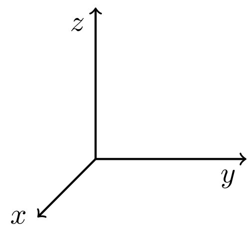
Example 22.10. *Sketch the graph of $r^2 \cos^2 \theta + z^2 = 9$*



Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

22.3.2 Integration with Cylindrical Coordinates

Recall, Cylindrical Coordinates are just polar coordinates with a __________!



Cartesian→Cylindrical:
$(x, y, z) \to (r, \theta, z)$:
$r^2 = x^2 + y^2$
$\theta = \arctan(\frac{y}{x})$
$z = z$

Cylindrical→Cartesian:
$(r, \theta, z) \to (x, y, z)$:
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

Example 22.11. *Which of the following regions represents the portion of the cylinder of height 4 and radius 3 above the 3rd quadrant of the xy plane?*

(a) $0 \le r \le 3, 0 \le z \le 4, 0 \le \theta \le \frac{\pi}{2}$
(b) $0 \le r \le 3, 0 \le z \le 4, \pi \le \theta \le \frac{3\pi}{2}$
(c) $1 \le r \le 4, 0 \le z \le 3, \pi \le \theta \le \frac{3\pi}{2}$
(d) $1 \le r \le 3, 0 \le z \le 4, 0 \le \theta \le \frac{\pi}{2}$

(e) I love cats!

Now let $f(r, \theta, z)$ be a continuous function defined on a region R where $R := \{(r, \theta, z) : a \le \theta \le b, g(\theta) \le r \le h(\theta), G(r, \theta) \le z \le H(r, \theta)\}$ where $g, h, G,$& H are continuous. Then the triple integral of f over R in cylindrical coordinates is
$\iiint_R f(r, \theta, z)dV =$

Example 22.12. *Which of the following is equivalent to* $\displaystyle\int_{-5}^{5}\int_{0}^{3}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} x\,dy\,dz\,dx$*?*

a) $\displaystyle\int_0^3\int_0^3\int_0^{\pi} r^2\cos\theta d\theta dz dr$
b) $\displaystyle\int_0^5\int_0^3\int_0^{\pi} r^2\cos\theta d\theta dz dr$
c) $\displaystyle\int_0^3\int_0^5\int_0^{2\pi} r\cos\theta d\theta dz dr$
d) $\displaystyle\int_0^5\int_0^3\int_0^{2\pi} r^2\cos\theta d\theta dz dr$
(e) More than one of the above.

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Example 22.13. *Which of the following is an iterated integral that will evaluate* $\iiint_E (x-y)dV$ *where* E *is the solid that lies between the cylinders* $x^2+y^2=1$ *and* $x^2+y^2=9$*, above the* xy*-plane, and below the plane* $z=y+4$*.*

a) $\displaystyle\int_0^{2\pi}\int_0^{y+4}\int_1^3 (x-y)rdrdzd\theta$

b) $\displaystyle\int_0^{2\pi}\int_1^3\int_0^{y+4} (x-y)rdzdrd\theta$

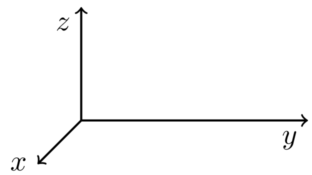
c) $\displaystyle\int_0^{2\pi}\int_0^{4+r\sin\theta}\int_1^3 (r\cos\theta-r\sin\theta)rdrdzd\theta$

d) $\displaystyle\int_0^{2\pi}\int_1^3\int_0^{4+r\sin\theta} (r\cos\theta-r\sin\theta)rdzdrd\theta$

e) none of the above

Example 22.14. *Set up an integral in cylindrical coordinates that calculates the volume of the region bounded by the paraboloid* $z=24-x^2-y^2$ *and the cone* $z=2\sqrt{x^2+y^2}$*.*

22.3.3 Spherical Coordinates



$\rho \geq$ __________,
__________ $\leq \theta \leq$ __________,
and __________ $\leq \phi \leq$ __________.

Converting Between Cartesian and Spherical Coordinates:

Cartesian to Spherical: $(x, y, z) \to (\rho, \theta, \phi)$:
$\rho =$

$\theta =$

$\phi =$

Spherical to Cartesian: $(\rho, \theta, \phi) \to (x, y, z)$:
$x =$

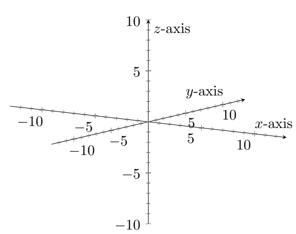
$y =$

$z =$

Important Note: Many physicists, engineers, scientists, and mathematicians reverse the roles of θ and ϕ. Some even use r instead of ρ (because ρ represents charge or mass density.). That is ϕ denotes the polar angle in the xy-plane, and θ denotes the angle from the positive z-axis. Our Calculus book uses the convention for θ to denote the same thing as it does in polar coordinates and it lists the order of spherical coordinates as (ρ, polar angle, angle from positive z-axis). Your physics class may switch the order of these coordinates. I am very sorry for the confusion and hope that one day we will all have the same convention (maybe change polar coordinates to have ϕ instead of θ?). In the meantime, I believe in your maturity to overcome this!

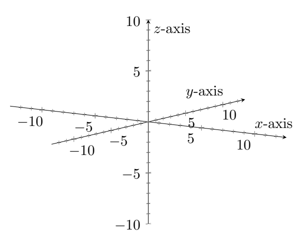
Example 22.15. *What are the Cartesian coordinates of the point with spherical coordinates* $(\rho, \theta, \phi) = (4, \pi, 0)$*?*

(a) (x, y, z) = (0, 0, -4)
(b) (x, y, z) = (0, 0, 4)
(c) (x, y, z) = (4, 0, 0)
(d) (x, y, z) = (-4, 0, 0)
(e) (x, y, z) = (0, 4, 0)



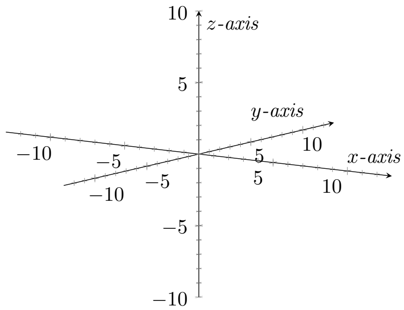
Example 22.16. *What are the spherical coordinates of the point with Cartesian coordinates* $(x, y, z) = (0, 3, 0)$*?*

(a)$(\rho, \theta, \phi) = (3, \frac{\pi}{2}, \pi)$
(b) $(\rho, \theta, \phi) = (3, \frac{-\pi}{2}, \pi)$
(c) $(\rho, \theta, \phi) = (3, \frac{\pi}{2}, \frac{\pi}{2})$
(d) $(\rho, \theta, \phi) = (3, \frac{-\pi}{2}, \frac{\pi}{2})$
(e) $(\rho, \theta, \phi) = (3, \pi, \frac{\pi}{2})$



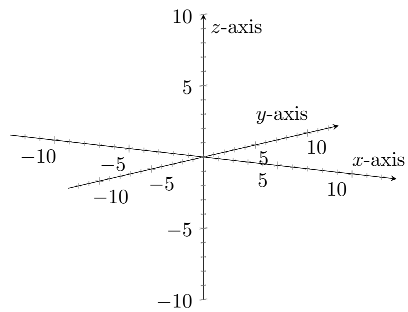
Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Example 22.17. *Sketch the graph of* $\rho = \sin(\phi)\cos\theta$



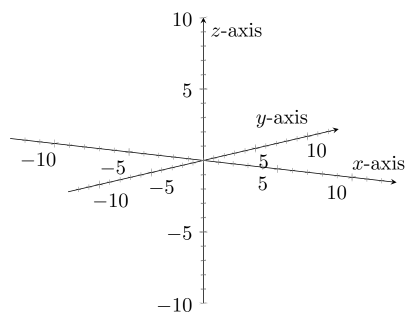
Example 22.18. *The graph of* $\rho = 5$ *is a*

(a) circle
(b) sphere
(c) cone
(d) plane
(e) cylinder



Example 22.19. *The graph of* $\phi = \frac{\pi}{4}$ *is a*

(a) circle
(b) sphere
(c) cone
(d) plane
(e) cylinder

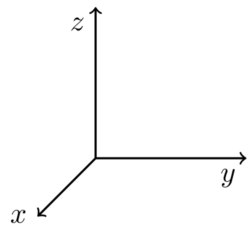


Example 22.20. *Is the graph of* $\theta = \dfrac{\pi}{4}$ *the same graph in spherical coordinates as it is in cylindrical?*

(a) yes *(b) no* *(c) I want to go home*

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

22.3.4 Integration with Spherical Coordinates



Cartesian→Spherical:
$(x, y, z) \to (\rho, \theta, \phi)$:
$\rho^2 = x^2 + y^2 + z^2$
$\theta = \arctan(\frac{y}{x})$
$\phi = \arccos(\frac{z}{\sqrt{x^2+y^2+z^2}})$

Spherical→Cartesian:
$(\rho, \theta, \phi) \to (x, y, z)$:
$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

Now let $f(\rho, \phi, \theta)$ be a continuous function defined on a region R where $R := \{(\rho, \phi, \theta) : a \leq \theta \leq b, g(\theta) \leq \phi \leq h(\theta), G(\phi, \theta) \leq \rho \leq H(\phi, \theta)\}$ where $g, h, G,$& H are continuous. Then the triple integral of f over R in spherical coordinates is
$\iiint_R f(\rho, \phi, \theta) dV =$

Example 22.21. *Which of the following integrals gives the volume of the unit sphere?*

a) $\displaystyle\int_0^{2\pi} \int_0^{2\pi} \int_0^1 d\rho d\phi d\theta$
b) $\displaystyle\int_0^{2\pi} \int_0^{\pi} \int_0^1 d\rho d\phi d\theta$
c) $\displaystyle\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\phi d\rho d\phi d\theta$
d) $\displaystyle\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\phi d\rho d\theta d\phi$
e) $\displaystyle\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho d\rho d\theta d\phi$

Example 22.22. *Evaluate* $\iiint_R e^{-(x^2+y^2+z^2)^{\frac{3}{2}}} dV$ *where R is the unit sphere.*

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Example 22.23. *Set up an integral to find the volume of the snow cone made from the region inside the cone* $\phi = \frac{\pi}{3}$ *and inside the sphere* $\rho = 5$.

Example 22.24. *Set up an integral to find the volume of the snow cone made from the region inside the cone* $z = \sqrt{x^2 + y^2}$ *and inside the sphere* $\rho = 5$.

Example 22.25. *Set up an integral to find the volume of the region bounded by the plane* $z = 25$ *and the paraboloid* $z = x^2 + y^2$*.*

22.3.5 ICE Integration Using Cylindrical and Spherical Coordinates

1. Set up an integral to find the volume of the solid under the surface $z = xy$, above the xy-plane, and within the cylinder $x^2 + y^2 = 4x$

2. Set up an integral that represents the volume above the cone $z = \sqrt{2(x^2 + y^2)}$ and below the sphere $x^2 + y^2 + z^2 = z$.

23 Changing Variables in Integration

Ever wonder why when we change coordinate systems and then integrate we need to change:

Polar: Cylindrical: Spherical:

These are derived through a change of variable formula. In fact, we are very comfortable with changing variables, we discussed this when we did rotations and translations of axes AND we do this every time we integrate using __________________!!

$$\int_a^b f(g(x))g'(x)dx =$$

Note: If g is a one-to-one (that is, if g has an inverse function), then starting with $\int_a^b f(x)dx$, we can transform $\int_a^b f(x)dx =$

In this section, we will be writing x, y, and z in terms of other new variables and the goal of these "transformations" is to make our lives easier when we integrate!

Example 23.1. *If we write x and y as functions of new variables u and v so if $x = 2u + 3v$ and $y = u - v$, the following points in the $u - v$ plane can be transformed into the xy plane by:*

$(0, 0) \rightarrow$ $(3, 0) \rightarrow$ $(3, 1) \rightarrow$ $(0, 1) \rightarrow$

Example 23.2. *Find the transformation from the xy plane into the uv plane for $u = 2x - 3y$ and $v = y - x$. In other words, find equations for x and y in terms of u and v.*

23.1 The Jacobean

Recall the determinant of $\begin{vmatrix} a & b \\ c & d \end{vmatrix} =$

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} =$

Definition: Given a transformation $T : x = g(u, v), y = h(u, v)$ where g and h are differentiable on a region in the uv space, the *Jacobian determinant* or the *Jacobian* of the transformation is $J(u, v) = \dfrac{\partial(x, y)}{\partial(u, v)} =$

If we have 3 variables, $J(u, v, w) = \dfrac{\partial(x, y, z)}{\partial(u, v, w)} =$

Example 23.3. *Compute the Jacobian for the transformation in Example 23.2.*

Theorem: If T is a one to one transformation which maps a bounded region S in the uv onto the bounded region R in the xy, if G is of the form $G(u,v) = (x(u,v), y(u,v))$, then $\iint_R f(x,y)dxdy =$

And we have the Triple Integral version: If $G(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$, then $\iiint_R f(x,y,z)dxdydz =$

Example 23.4. *Verify that $dV = rdzdrd\theta$. [We can also show $dV = \rho^2 \sin\phi d\rho d\phi d\theta$, but this is messier.]*

Example 23.5. *Sometimes change of variables is useful in solving an integral which seems tame, but has a "nasty" region: Consider* $\iint_R y^2 dA$ *where* R *is the region in the first quadrant bounded by the parabolas* $x = y^2$, $x = y^2 - 9$,$x = 4 - y^2$, *and* $x = 16 - y^2$. *Use change of variables to convert* R *into a rectangular region and set up the new integral.*

23.2 ICE Change of Variables

1. Evaluate the double integral $\iint_R \sin(x-y)\cos(x+y)dA$ where R is the triangle with vertices $(0,0), (\pi,-\pi)$, and (π,π).

24 Vector Fields

24.1 Introduction to Vector Fields

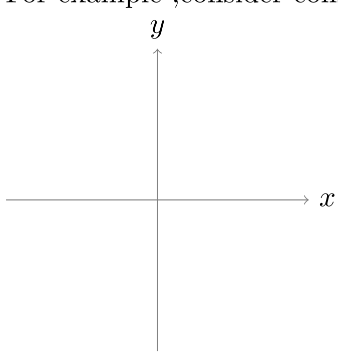
What is a vector field?
A vector field is function whose domain is the set $\mathbb{R}^2$ or $\mathbb{R}^3$ and whose range is a set of ________.
More formally,

Definition: Given f and g on a region $R \subseteq \mathbb{R}^2$, a *vector field* in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns each point in R a vector ______________________. So we can write:

Note: A vector field is continuous or differentiable on a region R if f and g are continuous or differentiable on R respectively.

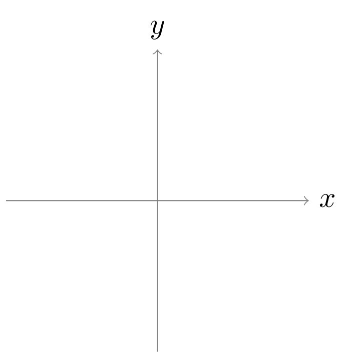
Key Concept: A vector field cannot be represented by a single curve or surface, instead we plot a sample of vectors to give us the general idea of the appearance of the vector field.

For example ,consider consider the vector field defined by $\mathbf{F} =< x, y >$



Idea: for each point $P = (x, y)$ plot a vector with its tail at P equal to the value of $\mathbf{F}(x, y)$. So the length of our vector is ________.

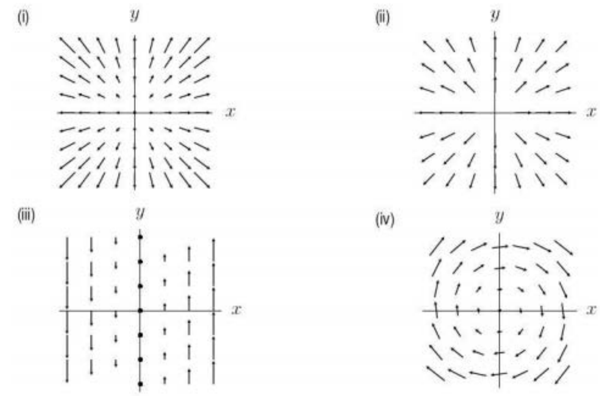
Example 24.1. *Sketch the vector field defined by* $\mathbf{F} =< 1 - y, 0 >$ *for* $|y| \leq 1$.



Example 24.2. *Match the vector fields with the appropriate graphs. Let* $\mathbf{r} = < x, y >$.

1. $\mathbf{F_1} = \dfrac{\mathbf{r}}{||\mathbf{r}||}$
2. $\mathbf{F_2} = \mathbf{r}$
3. $\mathbf{F_3} = y\mathbf{i} - x\mathbf{j}$
4. $\mathbf{F_4} = x\mathbf{j}$

(a) 1 and III, 2 and I, 3 and IV, 4 and II
(b) 1 and IV, 2 and I, 3 and II, 4 and III
(c) 1 and II, 2 and I, 3 and IV, 4 and III
(d) 1 and II, 2 and IV, 3 and I, 4 and III
(e) 1 and I, 2 and II, 3 and IV, 4 and III



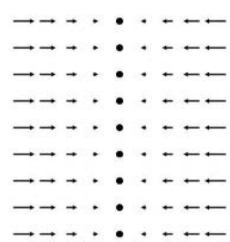
Note: Vector Fields come up in electric fields, magnetic fields, force fields, and gravitational fields. In our class, we will only consider fields which are independent of time and we call these fields, __________ vector fields.

We also have "scalar fields" which is just a fancy name for a function of space. These fields associate a number with a position in space. Examples of scalar fields include temperatures of plates and electrostatic potential.

We have already discussed one type of vector field, called __________ fields. The $\nabla f(x, y, z)$ points in the direction of greatest __________ of $f(x, y, z)$.

Example 24.3. *The figure shows the vector field* $\mathbf{F} = \nabla f$. *Which of the following are possible choices for* $f(x, y)$*? [Hint: Find* ∇f*.]*

a) x^2
b) $-x^2$
c) $-2x$
d) $-y^2$



Definition: A vector field, $\mathbf{F}$, that is the gradient of a scalar function is called a __________ vector field. In other words, there exists a scalar function ϕ such that __________.
We call ϕ a ____________________ function for $\mathbf{F}$.

We will talk more about how we can test vector fields to find out whether or not they are conservative in a later section. Woo!

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

24.2 Curl and Divergence

Definition: Let $\mathbf{F} =< f, g, h >$ where f, g, & h are functions of x, y, & z for which the first partial derivatives exist. Then we define the *divergence* of $\mathbf{F}$ of __________$\mathbf{F}$=

and the *curl* of $\mathbf{F}$ or __________$\mathbf{F}$ =

We call the divergence of $\mathbf{F}$ a __________ derivative. Using a slight but helpful abuse of notation, we see that div$\mathbf{F}$ = __________
If we think of $\mathbf{F}$ as the vector field of a flowing liquid, then div$\mathbf{F}$ represents the net rate of change of the mass of the liquid flowing from the point (x, y, z). We will learn more about divergence later.

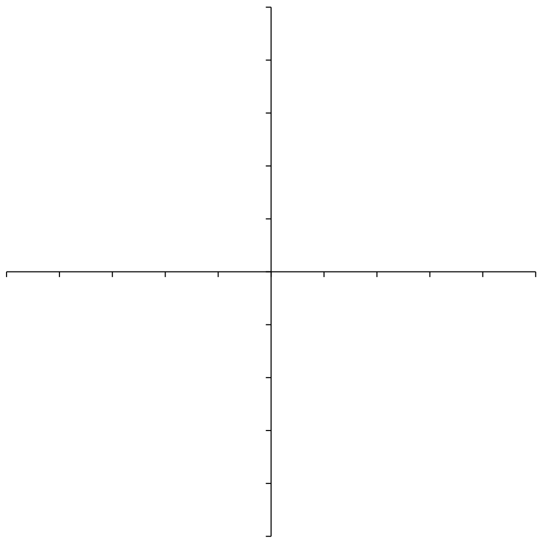
We call the curl of $\mathbf{F}$ a __________ derivative. Notice curl$\mathbf{F}$ = __________. curl$\mathbf{F} = \mathbf{0}$ if and only if $\mathbf{F}$ is conservative –aka if $\mathbf{F}$ is "nice." Later we will talk about how it measures the net counter clockwise circulation around the axis that points in the direction of the curl.

Example 24.4. *Find the divergence and curl of* $\mathbf{F}$ *for* $\mathbf{F}(x, y, z) =< \cos x, \sin y, 3 >$

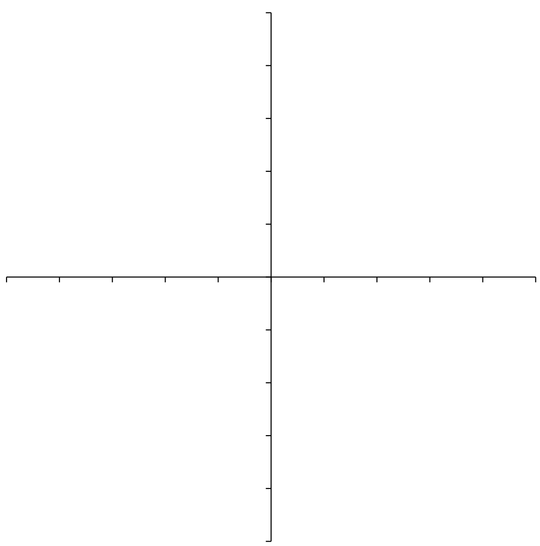
24.3 ICE Vector Fields

Sketch each of the following vector fields.

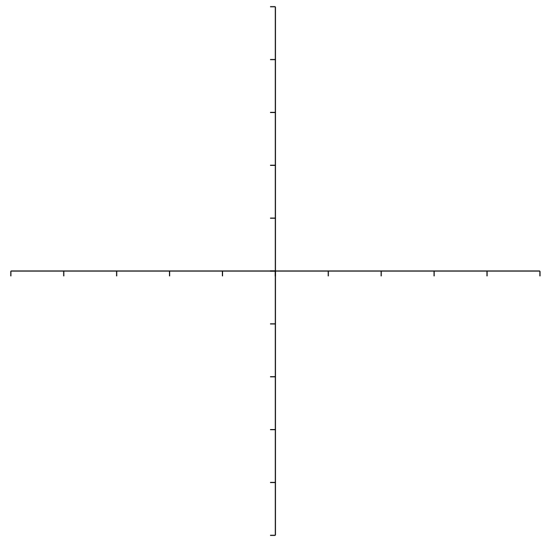
1. $-x\mathbf{i} - y\mathbf{j}$



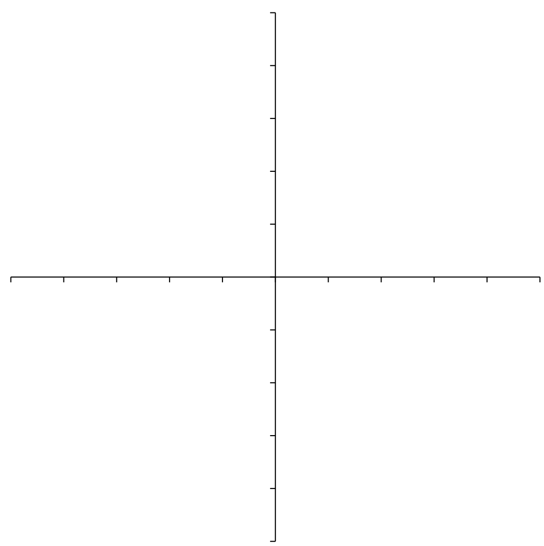
2. $y\mathbf{i} - x\mathbf{j}$



3. $\dfrac{-x\mathbf{i} - y\mathbf{j}}{\sqrt{x^2 + y^2}}$

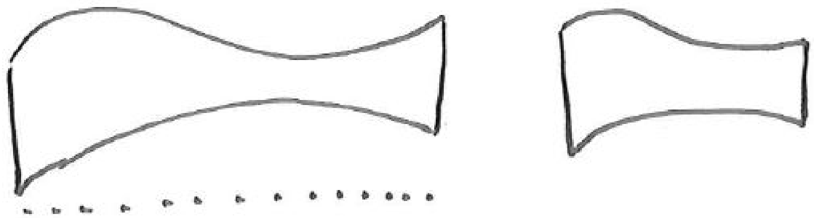


4. $y^2\mathbf{i} + x^2\mathbf{j}$



25 Line Integrals

When we evaluate $\displaystyle\int_a^b f(x)dx$, we are integrating over __________. We can generalize our integral by integrating over a __________ or __________ instead of our __________. This gives us something called a __________ integral. Although your text is correct when it says that they would be more accurately named __________ integrals.



Definition: If $f(x, y)$ is defined on a smooth curve C given by $x = x(t), y = y(t), a \le t \le b$, then the line integral (**with respect to arc length**) of f along a curve c is
$\int_C f(x, y)ds =$

The 3-space version: $\int_C f(x, y, z)ds =$

Definition: *The line integral of f along C* ***with respect to*** x is given by $\int_C f(x, y)dx =$

The line integral of f along C ***with respect to y*** is given by $\int_C f(x, y)dy =$

Often the line integrals **with respect to x and y** occur together we usually write
$\int_C P(x, y)dx + \int_C Q(x, y)dy =$

Note: The choice of parametrization will not change the path of our curve which means our line integral will not change. And if we are taking a line integral with respect to __________ __________, switching the direction of the curve won't change the value of the line integral.

Example 25.1. *Evaluate* $\int_C xyds$, *where* $C : \mathbf{r}(t) = 5t\mathbf{i} + 4t\mathbf{j}$, $0 \le t \le 1$.

Example 25.2. *a) Evaluate* $\int_C (x+y)ds$ *where* C *is the right half of the unit circle. Assume we move along the curve in the counter-clockwise direction.*

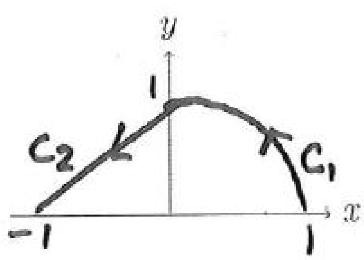
b) Evaluate $\int_C (x+y)ds$ *using a different parametrization.*

c) Evaluate $\int_C (x+y)ds$ *where* C *is the same, but assume we move along the curve in the clockwise direction.*

Example 25.3. *Evaluate* $\int_C xy^2dx + 2x^2ydy$, *where* C *is the line segment from* $(0,1)$ *to* $(4,3)$.

Note: Sometimes we need to break up our interval if our curve formula changes.

Example 25.4. *Set up* $\int_C (x+y)ds$ *where* $C = C_1 \cup C_2$ *is the curve shown in the figure.*



25.1 Line Integrals of Vector Fields

The motivation to evaluate line integrals of vector fields is ____________! Recall, we used a dot product to compute the work done by a constant force $\mathbf{F}$ from moving an object from point P to Q is $W =$ ____________ where $\mathbf{D}$ is the displacement vector ____________.

Now suppose we have a continuous force field (vector field) $\mathbf{F} =< f, g, h >$ and we wish to compute the work done by $\mathbf{F}$ in moving a particle along a smooth curve $C : \mathbf{r}(t) =< P(t), Q(t), R(t) >$ for $a \leq t \leq b$. When we partition our curve into subintervals and make our subintervals, Δs_i super small, then as our particle moves along our curve it moves approximately in the direction of the unit tangent vector of our curve at the point along the partition. Thus the work is approximately $\mathbf{F} \cdot \mathbf{T} \Delta s$.

Thus the work done is $\int_C \mathbf{F} \cdot \mathbf{T} ds =$
which is also equal to

Definition: Let $\mathbf{F} =< f, g, h >$ be a continuous vector field, then the *line integral of* $\mathbf{F}$ *along* C (where C is given by the vector function $\mathbf{r}(t)$ for $a \leq t \leq b$ is

Example 25.5. *A particle travels along the helix* C *given by* $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 2t\mathbf{k}$ *and is subject to the force* $\mathbf{F}(x, y, z) = x\mathbf{i} + z\mathbf{j} - xy\mathbf{k}$. *Find the total work done on the particle by the force for* $0 \leq t \leq 3\pi$.

25.2 ICE Line Integrals

1. Evaluate $\int_C xyds$ where C is the circle with radius 2 on the oriented path from $(2,0)$ to $(0,2)$.

2. Evaluate $\int_C xyds$ where C is the circle with radius 2 on the oriented path from $(0,2)$ to $(2,0)$.

3. What does this tell you about the relationship between $\int_C fds$ and $\int_{-C} fds$?

4. Evaluate $\int_C xyds$ where C is the line from $(2,0)$ to $(0,2)$.

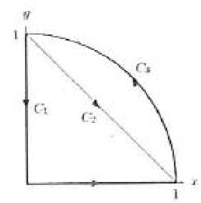
26 The Fundamental Theorem of Line Integrals

Remember the Grand 'ol Fundamental Theorem of Calculus?

Fundamental Theorem of Line Integrals: If C is a *piecewise* smooth curve given by $\vec{r}(t)$ on $a \leq t \leq b$ and f is a differentiable function on C with gradient ∇f, then
$\int_C \nabla f \cdot d\mathbf{r} =$

Example 26.1. *The line integral of* $\mathbf{F} = \nabla f$ *along one of the paths shown below is different from the integral along the other two. Which is the odd one out?*

a) C1
b) C2
c) C3



Key Ideas: If we are given $\int_C f d\mathbf{r}$, we can't necessarily use our theorem (we need __________.... but this theorem gives us the motivation to study the next few things...

26.1 Curve Talk with Dr. Harsy: Curve Adjectives:

Positively Oriented Curve: Simple Curve: Closed Curve:

Definition: A set D is *connected* if any two points in D can be connected by a piecewise curve that stays within D. D is __________ __________ if every simple path in D encloses points only in D.

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Definition: Given a connected set D We say $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ is *independent of path* in D if for any two points, say A & B in D, the line integral will have the same value for any path from A to B that stays in D AND is positively oriented.

AKA: The parameterization __________ __________ __________!!
So "path independent vector fields" have the property that for any two points, A and B, the line integral from A to B is independent of path.

Example 26.2. *Suppose C is a curve that starts and ends at the same point (called a closed curve). So* $\mathbf{r}(\mathbf{a}) = \mathbf{r}(\mathbf{b})$ *Then* $\int_C \nabla f \cdot d\mathbf{r} =$

Theorem: If vector field $\mathbf{F}$ is continuous on an open connected set D, then $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ is independent of path in D if and only if $\mathbf{F} =$ __________ for some scalar function f.

Recall we call such a scalar function f a ____________________ function for $\mathbf{F}$. And a vector field, $\mathbf{F}$, that is the gradient of a scalar function is called a ____________________ vector field.
Thus..
Theorem: The following are equivalent:

- $\mathbf{F}$ is a ____________________ vector field.
- $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ is
- $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = 0$ for any __________ path C.

Example 26.3. *A gradient field is path independent.*
a) True, and I am very confident!
b) True, but I am not very confident...
c) False, and I am very confident!
d) False, but I am not very confident...

Example 26.4. *Path independent vector fields are gradient fields.*
a) True, and I am very confident!
b) True, but I am not very confident...
c) False, and I am very confident!
d) False, but I am not very confident...
e) Snacks please!

Wouldn't it be nice if we could tell whether or not a vector field was a gradient/conservative/path independent vector field?

We would need to find a function, say ϕ such taht $F = \nabla\phi$. If we had this then $\int_c \mathbf{F}\cdot d\mathbf{r} =$

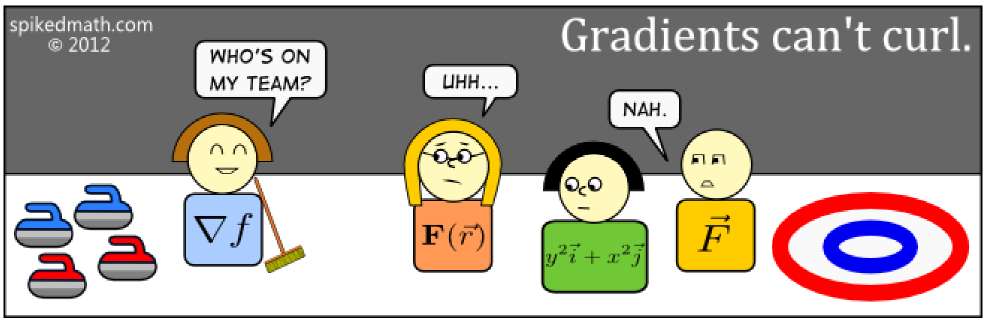
Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

26.2 Testing For Conservative Vector Fields

Testing for Conservative Vector Fields Theorem: Let $\mathbf{F} =< f, g, h >$ be a continuous vector field defined on a connected set D, then $\mathbf{F}$ is a conservative vector field (aka: we can find a ________________ function ________ s.t. $\mathbf{F} =$ ________) if and only if

In other words, ________$\mathbf{F} =$ ________

Note: If we have a vector field in $\mathbb{R}^2$, we only need:



Example 26.5. *Determine which vector fields are conservative.*
a) $\mathbf{F}(x, y) = e^{2y}\mathbf{i} + (1 + 2xe^{2y})\mathbf{j}$

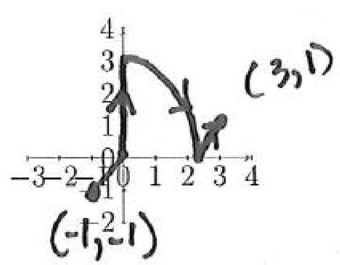
b)$\mathbf{F}(x, y, z) =< 2xy - z^2, x^2 + 2z, 2y - 2xz >$

c) $\mathbf{F}(x, y, z) = xy^2z^3\mathbf{i} + 2x^2yz^3\mathbf{j} + 3x^2y^2z^2\mathbf{k}$

26.3 Finding Potential Functions

Example 26.6. *Find a potential function for* $\mathbf{F} = e^{2y}\mathbf{i} + (1 + 2xe^{2y})\mathbf{j}$.

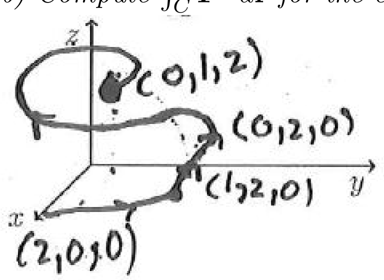
Example 26.7. *Evaluate* $\int_C (e^{2y}\mathbf{i} + (1 + 2xe^{2y})\mathbf{j}) \cdot d\mathbf{r}$ *where* C *is the curve below.*



Example 26.8. *Consider* $\mathbf{F} = < 2xy - z^2, x^2 + 2z, 2y - 2xz >$.

a) Find a potential function for $\mathbf{F}$. *What do we need to check first?*

b) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ *for the curve shown below.*

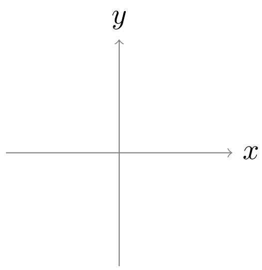


27 Curl and Divergence Revisted

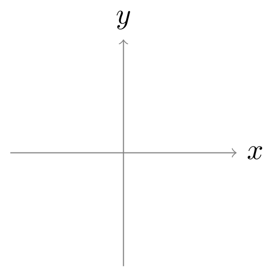
Recall if $\mathbf{F} =< f, g, h >$, curl $\mathbf{F} = < h_y - g_z, f_z - h_x, g_x - f_y >$ and if $\mathbf{F} \in \mathbb{R}^2$, say $\mathbf{F} =< P, Q, 0 >$, curl$\mathbf{F} =< 0, 0, Q_x - P_y >$. So if $Q_x - P_y = 0$, $\mathbf{F}$ is conservative. Otherwise $Q_x - P_y$ measures how "not conservative" $\mathbf{F}$ is. The following examples are going to help us visualize what is happening when we consider $Q_x - P_y$ for $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$.

First, let's look at: $\mathbf{F}(x, y, z) = Q\mathbf{j}$

If $Q_x < 0$:

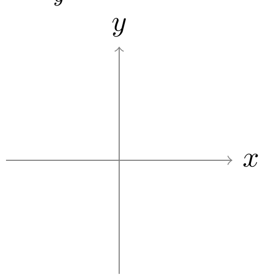


If $Q_x > 0$:

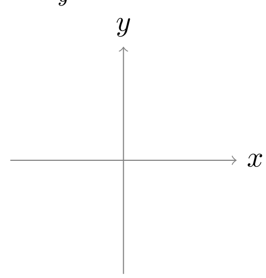


Now consider $\mathbf{F}(x, y, z) = P\mathbf{i}$

If $P_y < 0$:



If $P_y > 0$:



If $Q_x - P_y > 0$, our rotation is ____________________

If $Q_x - P_y < 0$, our rotation is ____________________

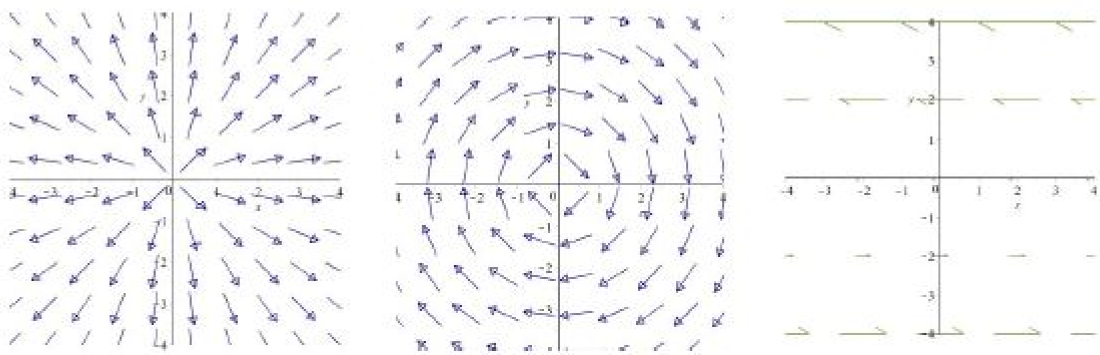
If $Q_x - P_y = 0$, our rotation is ____________________

So $Q_x - P_y$ measures the tendency to rotate counterclockwise!

Since $Q_x - P_y$ is determined by the __________ of a vector field, __________$\mathbf{F}$ measures the net counterclockwise circulation of a vector field.

Example 27.1. *For each field use the sketch to decide whether the curl at the origin points up down or is the zero vector. Then check your answer using the coordinate definition of curl.*

a) $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$ *b)* $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$ *c)*$\mathbf{F}(x, y) = -(y + 1)\mathbf{i}$

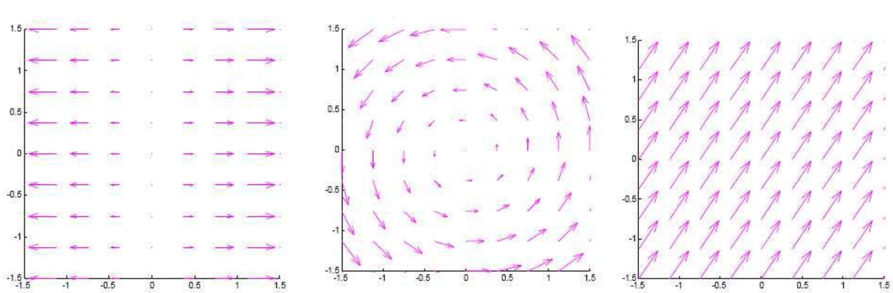


27.1 More About Vector Fields

Let's go back to the fundamental theorem of line integrals. In order to use this, we want a __________ vector field. In other words, we wonder whether $\mathbf{F}$ is a gradient field or whether of not it is path independent or whether or not it has rotation.

Recall, when we have a gradient field, the vectors are always __________ to the contours.

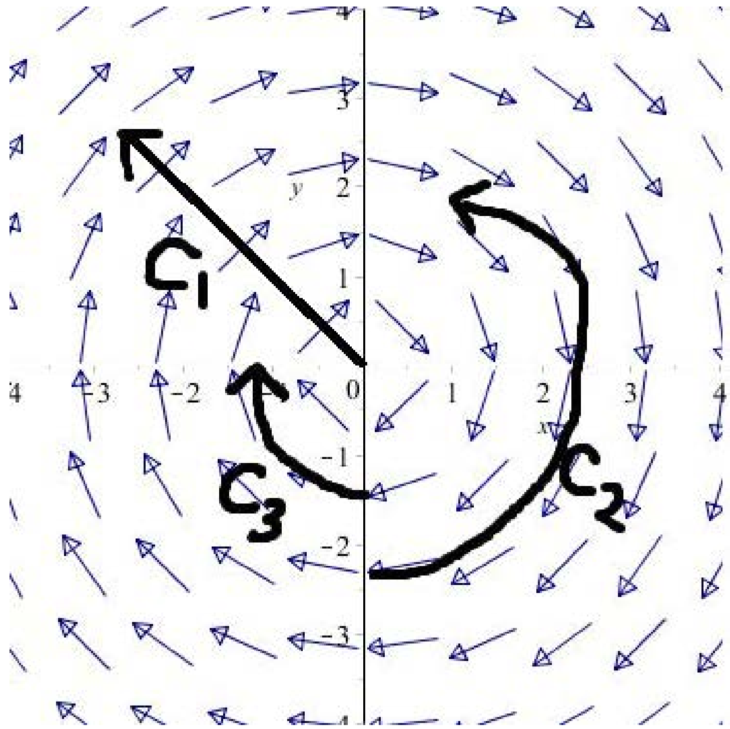
Example 27.2. *Which of the following vector fields cannot be a gradient vector field?*



a b

Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Example 27.3. *The figure below shows a path of a vector field* $\mathbf{F}$ *along with 3 curves* C_1, C_2, C_3*. Remember* $\int_C \mathbf{F} d\mathbf{r}$ *is calculating the work done in moving a particle through a force field along curve* C*.*

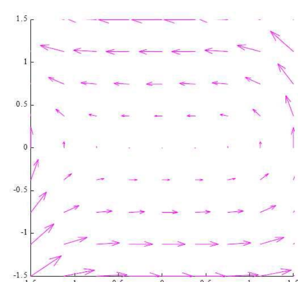


a) What is the sign of $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$?

b) What is the sign of $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$?

c) What is the sign of $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$?

Example 27.4. *Which of the following explains why this vector field is not a gradient vector field?*

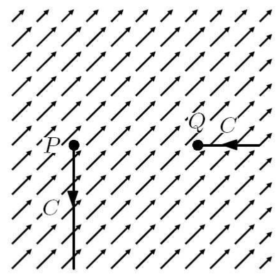


(a) The line integral from $(-1, 1)$ to $(1, 1)$ is negative.
(b) The circulation around a circle centered at the origin is zero.
(c) The circulation around a circle centered at the origin is not zero.
(d) None of the above

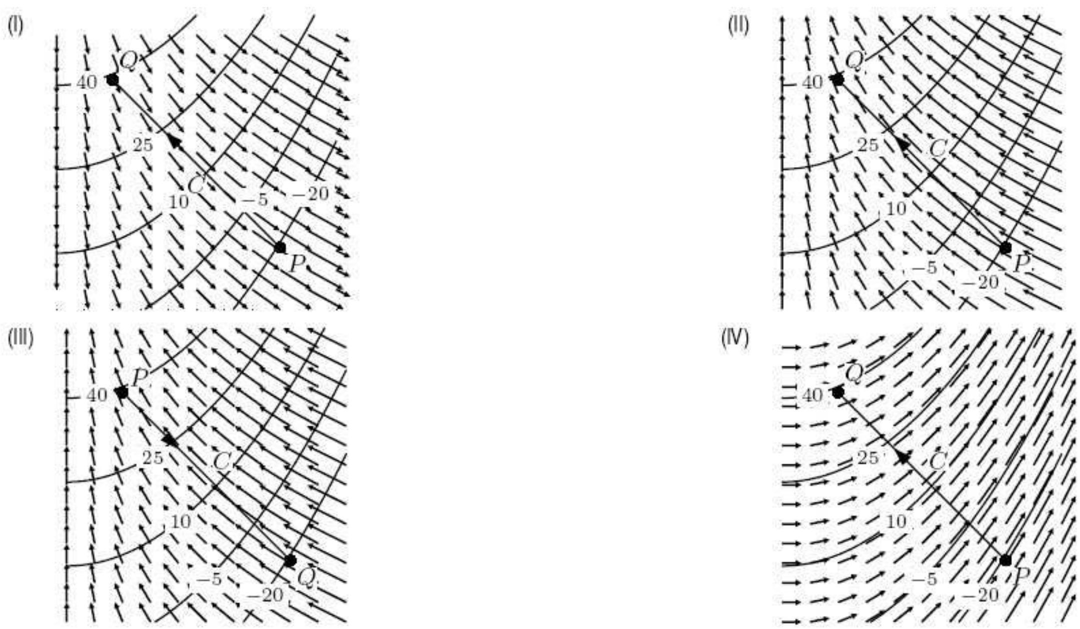
Multiple choice questions taken or modified from http://mathquest.carroll.edu/mvcalc.html.

Example 27.5. *The figure below shows the vector field ∇f, where f is continuously differentiable in the whole plane. The two ends of an oriented curve C from P to Q are shown, but the middle portion of the path is outside the viewing window. The line integral $\int_C \nabla f \cdot d\mathbf{r}$ is*

(a) Positive
(b) Negative
(c) Zero
(d) Can't tell without further information



Example 27.6. *Which of the diagrams contain all three of the following: a contour diagram of a function f, the vector field ∇f of the same function, and an oriented path C from P to Q with $\int_c \nabla F d\mathbf{r} = 60$?*



28 Green's Theorem

The rest of Calc III will be introducing different variations of our wonderful fundamental theorem of calculus. All of these generalizations come up in physic topics like electricity, magnetism, and fluid flow. So far we have discussed the fundamental theorem of line integrals. What did we need to have in order to use the fundamental theorem of line integrals?

We now discuss a variation of the Fundamental Theorem of Line Integrals in which we don't need our vector field to be ____________________.

Green's Theorem: Let C be a positively oriented piecewise smooth, simple, closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region containing D, then

Key: Green's Theorem can replace a difficult line integral by an easier double integral or vise versa.

Notation: When our curve is closed, we can write $\int_c$ as __________

If **F** is conservative, Green's Theorem says that the line integral along a closed curve is 0.
Green's Theorem is like the FTC because it relates an integral of a "derivative" on a region to its "antiderivative" on a boundary.

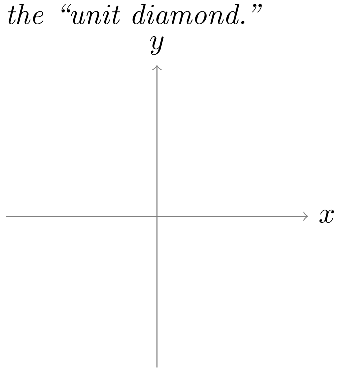
Example 28.1. *Evaluate the line integral* $\int_c ydx - xdy$ *where C is the unit circle centered at the origin directly without using Green's Theorem.*

Directly:

Example 28.2. *Evaluate the line integral* $\int_c y dx - x dy$ *where* C *is the unit circle centered at the origin using Green's Theorem.*

Green's:

Example 28.3. *Evaluate the line integral* $\int_c (x^4 + 2y)dx + (5x + \sin y)dy$ *where* C *is boundary of the "unit diamond."*



Notice, Green Theorem applies to a positively oriented piecewise smooth, simple, **closed** curve, regardless if our vector field is ____________________!!

28.1 ICE Green's Theorem

1. Evaluate $\oint \sqrt{1+x^3}dx + 2xydy$ where C is the positively oriented triangle with vertices $(0,0), (1,0)$, and $(0,3)$. Use Green's Theorem.

2. Evaluate $\oint \sqrt{1+x^3}dx+2xydy$ where C is the positively oriented square with vertices $(0,0), (2,0), (0,2)$ and $(2,2)$. Use Green's Theorem.

Turn over for more questions...

3. Evaluate $\oint \sqrt{1+x^3}dx + 2xydy$ where C is the line segment from $(1, 0)$ to $(0, 3)$.

4. Use Green's Theorem to calculate the line integral $\int_C xy^2dx - x^2ydy$ where C consists of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-1, 1)$.

29 Parametric Surfaces

Just like in the 2-dimensional case, in higher dimensions, it can be beneficial to write equations using parameters.

Given a surface $z = G(x, y)$, we can always parameterize it by

$x =$ $\qquad\qquad\qquad\qquad$ $y =$ $\qquad\qquad\qquad\qquad$ $z =$

We can also generalize this. If the surface is written as $x = G(y, z)$, we can parameterize it by
$x =$ $\qquad\qquad\qquad\qquad$ $y =$ $\qquad\qquad\qquad\qquad$ $z =$

Unfortunately this can only work if we can write x, y, or z as an explicit function of the other variables and sometimes we can't do this which means we turn to different parameterizations.

Example 29.1. *Parameterize the surface represented by the top half of the cone $z^2 = 3x^2 + 3y^2$.*
a) Note that since we only want the top half, we can write z as an explicit function of x and y.

b) Find another representation for this surface.

Example 29.2. *Parameterize the sphere $x^2 + y^2 + z^2 = 4$.*

29.1 Recognizing Surfaces From Parametric Equations

Match the following equations numbered 1-6 with the graphs on the next page. In each case determine what the grid curves are when u or v are constant. Eliminate the parameters where possible to give a rectangular equation for each surface.

1. $\mathbf{r}(u, v) = < \sin u, \cos u, v >$
   If u is constant, the grid curves are:

   If v is constant, the grid curves are:
   The surface number is:

2. $\mathbf{r}(u, v) = < \sin u, v, \cos u >$
   If u is constant, the grid curves are:

   If v is constant, the grid curves are:
   The surface number is:

3. $\mathbf{r}(u, v) = < v \sin u, v \cos u, v >$
   If u is constant, the grid curves are:

   If v is constant, the grid curves are:
   The surface number is:

4. $\mathbf{r}(u, v) = < v, v \cos u, v \sin u >$
   If u is constant, the grid curves are:

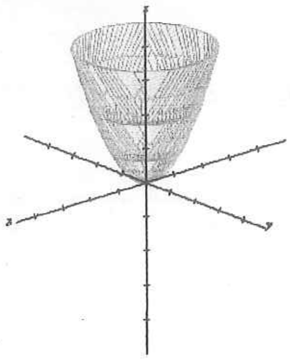
   If v is constant, the grid curves are:
   The surface number is:

5. $\mathbf{r}(u, v) = < v \sin u, v \cos u, v^2 >$
   If u is constant, the grid curves are:

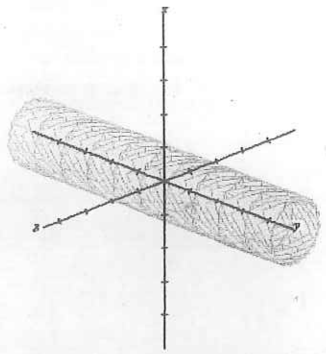
   If v is constant, the grid curves are:
   The surface number is:

6. $\mathbf{r}(u, v) = < v \sin u, v \cos u, u >$
   If u is constant, the grid curves are:

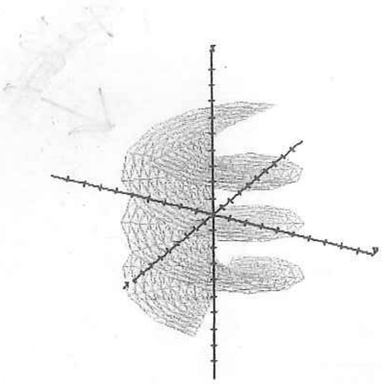
   If v is constant, the grid curves are:
   The surface number is:



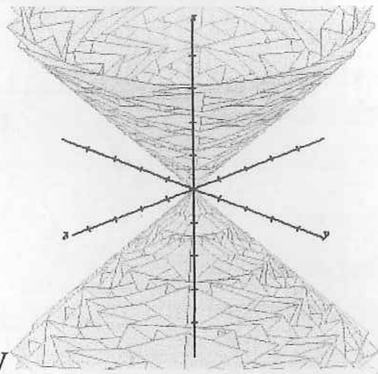
I



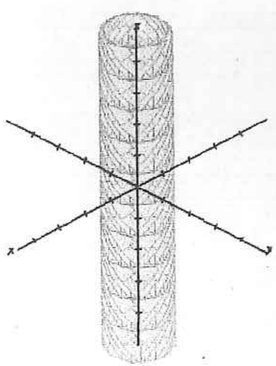
II



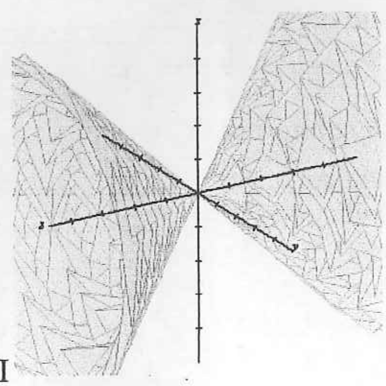
III



IV



V



VI

30 Surface Integrals

Recall from 21.4:

Definition: Given a smooth *parametric surface* S given by the equation $\mathbf{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$ (where S is covered just one as the parameters range through the domain of S), then the surface area of S is given by

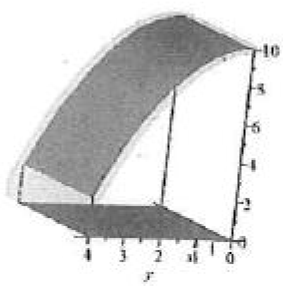
$$A(S) = \iint_D |r_u \times r_v| dA$$

Note $r_u =< x_u, y_u, z_u >$ and $r_v =< x_v, y_v, z_v >$.

Explicit Case: To find the Area of a surface represented explicitly by $z = g(x,y)$, we evaluate

$$A(S) = \iint_R \sqrt{1 + g_x^2 + g_y^2} dA$$

A line integral generalizes the ordinary definite integral by integrating over a ________. Now we discuss surface integrals which generalize ________ integrals by integrating over a ________.



Parameterizing by **u** and **v**:

$x =$ $\qquad\qquad$ $y =$ $\qquad\qquad$ $z =$

Surface:= $\mathbf{r}(u,v) =$

Parameterizing for explicit case when $z = g(x,y)$:

$x =$ $\qquad\qquad$ $y =$ $\qquad\qquad$ $z =$

Surface:= $\mathbf{r}(x,y) =$

$r_x =$

$r_y =$

$|r_x \times r_y| =$

Parametric Case: If a surface S is given by $\mathbf{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$, the **surface integral of a function f over S** is given by

$$\iint_G f(x,y,z)dS = \iint_R f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|dA$$

Explicit Case: Given a surface explicitly given by $z = g(x,y)$ with domain R. Let $f(x,y,z)$ be a function defined on S with continuous first-ordered partial derivatives, then *the surface integral of f over G* is

$$\iint_G f(x,y,z)dS = \iint_R f(x,y,g(x,y))\sqrt{1+(g_x)^2+(g_y)^2}dA$$

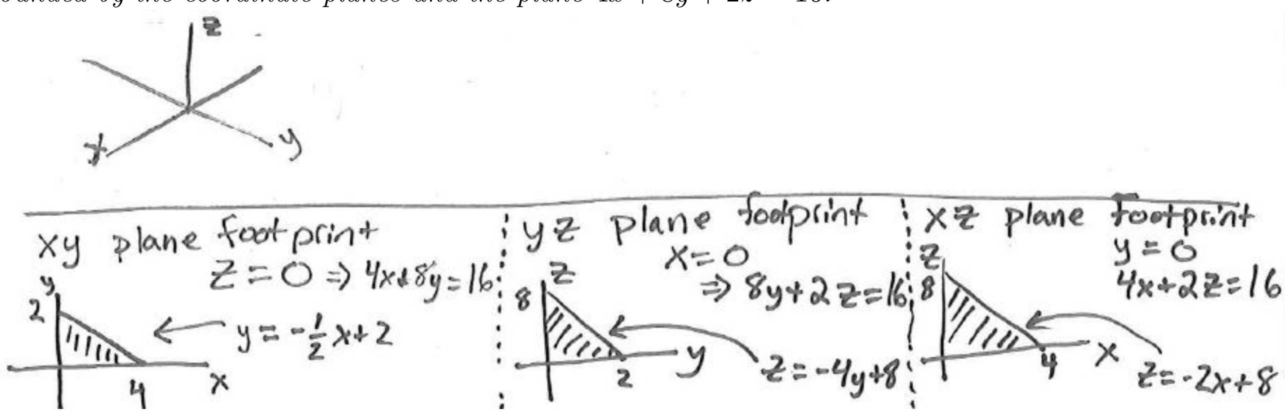
Often we think of $f(x,y,z) =$ ____________________ and can generalize it:

Example 30.1. *Evaluate the surface integral $\iint_G zdS$ where G is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$. (Note $z = x^2 + y^2$ is a 1 sided surface.)*

Example 30.2. *Set up the surface integral $\iint_G x^2 dS$ where G is the sphere with radius 1.*

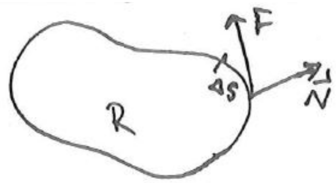
Key Technique: If G is a piece-wise surface, we must "break up" our surface into parts and integrate over each part of the surface.

Example 30.3. *Evaluate the surface integral* $\iint_G z dS$ *where* G *is the part of the tetrahedron bounded by the coordinate planes and the plane* $4x + 8y + 2z = 16$.

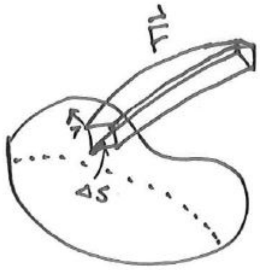


30.1 Surface Integrals of Vector Fields/ Flux

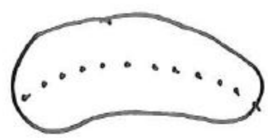
Definition: If $\mathbf{F}(x, y)$ denotes the velocity of the fluid at (x, y) in a region S (in xy plane) as it crosses its boundary curve C, the *flux* of of $\mathbf{F}$ is the net amount of fluid leaving S per unit of time.



Consider a region R enclosed by curve C.
So the **flux** (total amount of fluid leaving R) of $\mathbf{F}$ across the curve C is $\oint_C$



Now suppose we want calculate the flux or flow of a vector field $\mathbf{F}$ across a surface.
First we need to specify an orientation of our surface with a normal vector. That is, we need a 2-sided surface. (Note in Ex 1, $z = x^2 + y^2$ is a 1 sided surface.)



Surface Talk with Dr. Harsy:
For a closed surface, the "positive orientation" is ________
If the normal vector for a surface varies continuously, we say our surface is ________.

Definition: Given a continuous vector field $\mathbf{F} =< P, Q, R >$ defined on an oriented closed smooth surface $G = z = g(x, y)$ with domain R, the **surface integral of F over S** or **Flux Integral** is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS$$

Our normal vector $\mathbf{n}$ can be written as ________________ which means we also have that $\int_G \mathbf{F} \cdot \mathbf{n} dS =$

Parametric Case: For a surface given by $\mathbf{r}(u, v) =< x(u, v), y(u, v), z(u, v) >$, the **surface integral of F over S** is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dS$$

Example 30.4. *Calculate the flux of* $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ *for the following oriented surfaces.*
a) A unit square in the xy-plane oriented upwards.

b) A unit square in the yz-plane oriented towards the positive x-axis.

c) A unit square in the xz-plane oriented towards the negative y-axis.

Example 30.5. *Calculate the flux of* $\mathbf{F}(x, y, z) = (y)\mathbf{j}$ *across the part of the plane* $4x+8y+2z = 16$ *in the first octant. Note: Use Example 30.3.*

31 Stokes' Theorem

Example 31.1. *Consider the vector field given by* $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ *and the surface* S *that is the hemisphere* $x^2 + y^2 + z^2 = 1$, $z \geq 0$ *oriented upward.*

1) Evaluate curl$\mathbf{F}$

2) Sketch the surface S and a normal vector orienting the surface.

3) What is the boundary of S? Let's call it ∂S or C.

4) Parameterize C with *positive orientation.*

5) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. [Why must we use our parameterization of C?]

6) Now evaluate $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$. Recall, $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl}\mathbf{F} \cdot \mathbf{n} dS$.

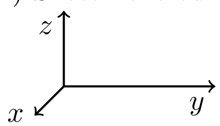
7) What do you notice about the answers you got in part 5 and 6?

This is an example of Stokes' Theorem:
Stokes' Theorem Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let **F** be a vector field whose components have continuous partial derivatives around S. Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S curl\mathbf{F} \cdot d\mathbf{S} = \iint_S curl\mathbf{F} \cdot \mathbf{n} dS$$

Example 31.2. *Let* $\mathbf{F} =< e^{-x}, e^x, e^z >$ *and let* C *be the boundary of the part of the plane* $2x + y + 2z = 2$ *in the first octant oriented counterclockwise as viewed from above. Use Stokes' Theorem to evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *using the following steps.*

1) Sketch the curve C, including orientation.



2) Why would calculating the line integral over C not be ideal?

3) Instead of calculating $\int_C \mathbf{F} \cdot d\mathbf{r}$, let's calculate:

First we need to describe a surface S whose boundary is C.

Then we determine an equation $z = g(x, y)$ and a domain D for that surface S:

Next we determine curl**F**

And also the normal vector **n**:

The we evaluate $\iint_S \text{curl}\mathbf{F} \cdot \mathbf{n} dS$. (Remember by Stokes' Theorem, this then equals $\int_C \mathbf{F} \cdot d\mathbf{r}$).

31.1 ICE Stokes' Theorem

1. Use Stokes' Theorem to evaluate the $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} =< 2y, -z, x >$ where C is the boundary of the square $|x| \leq 1$, $|y| \leq 1$ in the xy plane ($z = 0$). Do this by calculating $\iint_S \text{curl}\mathbf{F} \cdot \mathbf{n} dS$. Hint: Use the plane where the square is located as your surface. If I have a flat square in the x-y plane, what is $\mathbf{n}$?

2. Use Stokes' Theorem to evaluate the surface integral $\iint_S \text{curl}\mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xye^z\mathbf{k}$ and S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$. Note $\text{curl}\mathbf{F} =< xe^z - x, y - ye^z, 0 >$ so do this by calculating $\oint_C \mathbf{F} \cdot d\mathbf{r}$.
Hint: Use the trig identity: $\cos^2 t - \sin^2 t = \cos(2t)$.

32 Gauss's Divergence Theorem

32.1 Vector Forms of Green's Theorem

Curl and divergence operators allow us to rewrite Green's theorem. Suppose we have a plane region D with boundary curve C and function P and Q which satisfy Green's Theorem. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint Pdx + Qdy$.

If our curve is parameterized using arc length, say $x = x(s), y = y(s)$, then our tangent vector and normal vectors for our curve are given by:
$\mathbf{T} =$ $\mathbf{N} =$

Now $\oint_C \mathbf{F} \cdot \mathbf{n} ds =$

This is sometimes called Gauss's Divergence Theorem in the plane. Restated in terms of Flux, this means that the flux across C can be calculated as $\oint_C \mathbf{F} \cdot \mathbf{n} ds$.

32.2 Gauss's Divergence Theorem

We just discussed Gauss's Theorem in the Plane, now suppose we consider a surface that is the boundary of a solid region S. We denote the boundary of our solid S as ________.
Gauss's Theorem in the plane says that $\oint_{\partial S} \mathbf{F} \cdot \mathbf{n} ds = \iint_S \text{div}\mathbf{F} dA$

Gauss's Divergence Theorem: Let S be a simple solid with positive orientation and boundary ∂S. And let $\mathbf{F}$ be a vector field with continuous first partial derivatives around S. Then
$\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iint_{\partial S} \mathbf{F} \cdot \mathbf{n} dS =$

Note: The above integral calculates the flux of $\mathbf{F}$ across ________

Warning: The divergence theorem can only work with *simple solid* regions. If we want to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} =< xy, yz, zx >$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ and has upward orientation. Why can't we use the divergence theorem?

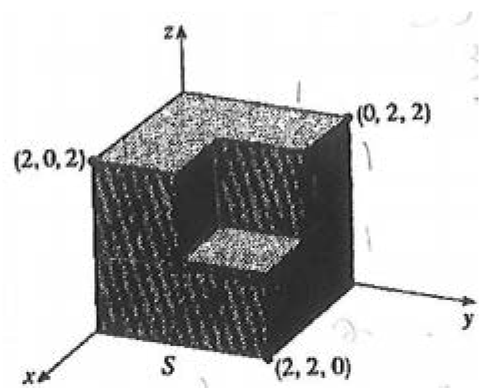
In this situation, we need to set up an integral to calculate this surface integral.

Example 32.1. *During our Surface Integral Notes, we calculated the flux of* $\mathbf{F}(x, y, z) = (y)\mathbf{j}$ *across the part of the plane* $4x + 8y + 2z = 16$ *in the first octant. Check our answer using Gauss's Divergence Theorem and use the fact that the volume of a tetrahedron is* $\frac{1}{3}h$*(area of the base).*

Example 32.2. *Consider* $\mathbf{F} = < \frac{xy^2}{2}, \frac{y^3}{6}, zx^2 >$ *over the surface* S *where* S *is the cylinder* $x^2 + y^2 = 1$ *capped by the planes* $z = \pm 1$*. What is the flux of* $\mathbf{F}$ *over* S*?*

Example 32.3. *Let S be any solid sphere. Consider the two 3-dimensional vector fields given by* $\mathbf{F}(x,y,z) =< 8x+3y, 5x+4z-2y, 9y^2 - \sin x + 7z >$ *and* $\mathbf{G}(x,y,z) =< 12y+8z, e^z + \sin x + 9y, xy^2e^{xy} + 4z >$*. Without performing any integration, show that* $\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iint_{\partial S} \mathbf{G} \cdot d\mathbf{S}$.

Example 32.4. *Determine* $\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} dS$ *where* $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *and* ∂S *is the outward oriented boundary of the surface shown in the figure below:*



Note: Example 32.4, if the divergence wasn't constant, the limits of integration would be very tough to set up. What would we have to do?

32.3 ICE Gauss' Divergence Theorem

1. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ (that is calculate the flux of $\mathbf{F}$) across S) for $\mathbf{F} =< 2x^3 + y^3, y^3 + z^3, 3y^2 z >$ and where S is the surface of the bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy plane.

   Now set up an integral to calculate this not using the divergence theorem.

2. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ (that is calculate the flux of $\mathbf{F}$) across S) for $\mathbf{F} =< x^3, y^3, z^3 >$ and where S is the surface of the solid bounded by the cylinder $1 = x^2 + y^2$ and the planes $z = 0$ and $z = 2$.

3. Note that the divergence theorem can only work with simple solid regions. If we want to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} =< xy, yz, zx >$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ and has upward orientation. Why can't we use the divergence theorem? Set up an integral to calculate this surface integral.

33 Vector Calculus Theorem Review

33.1 Line Integrals

33.1.1 Scalar Function Version:

Calculate the line integral of f over the curve c.This comes in many forms:

- $\int_c f ds$: parametrize with respect to arc length: $ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt$
- $\int_c f dx$: parametrize with respect to x: $dx = x'(t)dt$
- $\int_c f dy$: parametrize with respect to y: $dy = y'(t)dt$
- $\int_c f dz$: parametrize with respect to z: $dz = z'(t)dt$

Note $\oint$ means that the curve is closed. Steps for calculating Line Integrals:

1. Identify which form of a line integral you have.
2. Parameterize depending on whether you have a ds, dx, dy, dz or if the curve is already given to you.
3. Rewrite the integral only in terms of your parameter (usually t).

33.1.2 Vector Field Version:

The line integral (sometimes called the Flux Integral) of $\mathbf{F} =< P, Q, R >$ over the curve $c := \mathbf{r}(t)$ is given by

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_c (Pdx + Qdy + Rdz)$$

33.2 Fundamental Theorem of Line Integrals

This is sometimes called the Gradient Theorem for Line Integrals. If f is continuously differentiable and $c : [a, b] \to \mathbb{R}$ is a piecewise continuously differentiable path, then

$$\int_c \nabla f \cdot ds = f(c(b)) - f(c(a))$$

Note this means if the vector field is conservative, that is if $\mathbf{F} = \nabla f$ for some scalar f,

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_c \nabla f \cdot ds = f(c(b)) - f(c(a))$$

Steps to use Fundamental Theorem of Line Integrals:

1. Determine whether $\mathbf{F}$ is conservative. Often this is done by verifying that $\text{curl}\mathbf{F} = \mathbf{0}$
2. Find a potential function for $\mathbf{F}$. That is find, f such that $\mathbf{F} = \nabla f$
3. Apply the Fundamental Theorem of Line Integrals.

Note if $\mathbf{F}$ is conservative, then $\oint \mathbf{F} \cdot d\mathbf{r} = 0$.

33.3 Green's Theorem

Let c be a positively oriented piecewise smooth, simple, closed curve in the plane and let D be the region bounded by c. If $\mathbf{F} =< P, Q >$, then

$$\oint_c \mathbf{F} \cdot d\mathbf{r} = \oint_c (Pdx + Qdy) = \iint_D (Q_x - P_y) dA$$

Note this theorem allows us to calculate a difficult line integral (which may have multiple parameterizations) with a hopefully easier double integral or vise versa. Note if $\mathbf{F}$ is conservative, then $Q_x - P_y = 0$ so of course $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ (as we would expect from the Fundamental Theorem of Line Integrals.)
Steps to use Green's Theorem:

1. Make sure we have a positively oriented, piecewise smooth, simple, and closed curve.
2. Calculate $Q_x - P_y$.
3. Find the bounds for the *double* integral over the region D.
4. Calculate the double integral.

Note: If you want to take a difficult double integral and calculate the line integral,
Find $F =< P, Q >$ and parameterize the line integral with respect to the boundary of D.

33.4 Stokes' Theorem

Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S curl\mathbf{F} \cdot d\mathbf{S} = \iint_S curl\mathbf{F} \cdot \mathbf{n} dS$$

Note: Stokes' Theorem is a variation of Green's Theorem. It can replace a difficult line integral with a hopefully easier surface integral or vise versa.
Steps to use Stokes' Theorem:

1. Calculate the curl$\mathbf{F}$.
2. Find the normal vector $\mathbf{n}$ which depends on the parameterization of the surface (see Section 33.6 for more details)
3. Find the region of integration for the surface in order to calculate the double integral.

Note: This works well when you have a surface with multiple faces/sides which means you would need multiple parameterizations. On the other hand, if the curl$\mathbf{F}$ is very complex for the surface integral, you may decide to set up the line integral instead. Finally, of course if $\mathbf{F}$ is conservative, then curl$\mathbf{F} = \mathbf{0}$ which would mean that the value of the line integral/surface integral is 0 as expected for a closed curve C.

33.5 Gauss' Divergence Theorem

Let S be a simple solid with positive orientation and boundary ∂S and $\mathbf{F} =< P, Q, R >$ be a vector field with continuous first partial derivatives around S, then

$$\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iint_{\partial S} \mathbf{F} \cdot \mathbf{n} dS = \iiint_{S} div(\mathbf{F}) dV$$

Note: This theorem allows us to replace a difficult (possibly multi-faced) surface integral with a hopefully easier triple integral or vise versa.
Steps to Use Gauss' Divergence Theorem:

1. Check to make sure you have a simple solid contained by the surface.
2. Calculate div($\mathbf{F}$)
3. Find the region of integration for the solid in order to calculate the triple integral. You may need to use spherical or cylindrical coordinate systems.

33.6 Surface Integrals

33.6.1 Scalar Form of Surface Integrals:

Given a smooth surface, $S = \mathbf{r}(u, v) =< x(u, v), y(u, v), z(u, v) >$, then the surface integral of a function f over S is given by

$$\iint_G f(x, y, z) dS = \iint_R f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$$

Special Case: Explicit Functions: Given a surface explicitly given by $z = g(x, y)$ with domain R, then the surface integral of f over G is given by

$$\iint_G f(x, y, z) dS = \iint_R f(x, y, g(x, y)) \sqrt{1 + (g_x)^2 + (g_y)^2} dA$$

Note: This is a special case of the general parameterization since you can always parameterize an explicit function $z = g(x, y)$ in the following way:

$$x = x, \qquad y = y, \qquad z = g(x, y) \text{ with } |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + (g_x)^2 + (g_y)^2}$$

Note: We can generalize this. For example if $y = g(x, z)$ then

$$x = x, \qquad y = g(x, z), \qquad z = z \text{ with } |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{1 + (g_x)^2 + (g_z)^2}$$

So $\iint_G f(x, y, z) dS = \iint_R f(x, g(x, z), z) \sqrt{1 + (g_x)^2 + (g_z)^2} dA$.

Steps to calculate general scalar surface integrals:

1. Parameterize the surface. This is relatively straight forward with explicit surfaces. When you don't have an explicit surface often using other coordinate systems like spherical or cylindrical can help.
2. Calculate $|\mathbf{r}_u \times \mathbf{r}_v|$ or $\sqrt{1 + (g_x)^2 + (g_y)^2}$
3. Find the region of integration for the surface in order to calculate the double integral.

33.6.2 Vector Form of Surface Integrals:

Given a continuous vector field $\mathbf{F} =< P, Q, R >$ defined on an oriented closed smooth surface $S = \mathbf{r}(u, v) =< x(u, v), y(u, v), z(u, v) >$, the surface integral (sometimes called the Flux Integral) of $\mathbf{F}$ over S is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dS$$

Special Case: Explicit Functions: Given a continuous vector field $\mathbf{F} =< P, Q, R >$ defined on an oriented closed smooth surface $G = z = g(x, y)$ with domain R, the surface integral of $\mathbf{F}$ over S is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_R < P, Q, R > \cdot < -g_x, -g_y, 1 > dA$$

Note: $\mathbf{n} =< -g_x, -g_y, 1 >$ (or if $y = g(x, z)$, $\mathbf{n} =< -g_x, 1, -g_z >$).
Steps to calculate the surface integral:

1. Parameterize the surface if it isn't already given to you.
2. Write $\mathbf{F}$ in terms of the the parameterization.
3. Calculate $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$ or if the surface is explicitly defined, $\mathbf{n} =< -g_x, -g_y, 1 >$
4. Find the region of integration for the surface in order to calculate the double integral.

33.7 ICE -Theorem Review

For the following prompts identify what vector theorem would best apply to the problem.

Example 33.1. *Evaluate* $\int_c \sqrt{1+x^3}dx + 2xydy$ *where* c *is the boundary of a square. (Follow up question, what if* c *was a line segment?)*

(a) Fundamental Theorem of Line Integrals

(b) Green's Theorem

(c) Stokes' Theorem

(d) Gauss' Divergence Theorem

(e) Just calculate the line integral directly.

Example 33.2. *Calculate the Flux of* $\mathbf{F} =< xye^z, xy^2z^3, -ye^z >$ *across the surface of the box bounded by the coordinate planes and the planes* $x = 3, y = 2, z = 1$.

(a) Fundamental Theorem of Line Integrals

(b) Green's Theorem

(c) Stokes' Theorem

(d) Gauss' Divergence Theorem

(e) Just calculate the surface integral directly.

Example 33.3. *Evaluate* $\int_c \mathbf{F} \cdot d\mathbf{r}$ *for* $\mathbf{F} =< 4xy^2 - 2xy^3, 2x^4y - 3x^2y^2 + 4y^3 >$ *and* $\mathbf{r}(t) =< t + \sin(\pi t), 2t + \cos(\pi t) >$.

(a) Fundamental Theorem of Line Integrals

(b) Green's Theorem

(c) Stokes' Theorem

(d) Gauss' Divergence Theorem

(e) Just calculate the line integral directly.

Example 33.4. *Evaluate* $\iint_S \mathbf{F} \cdot d\mathbf{S}$ *where* S *is the surface* $z = x \sin y$, $0 \le x \le 2, 0 \le y \le \pi$.

(a) Fundamental Theorem of Line Integrals

(b) Green's Theorem

(c) Stokes' Theorem

(d) Gauss' Divergence Theorem

(e) Just calculate the surface integral directly.

Example 33.5. *Calculate* $\iint_S \mathbf{F} \cdot d\mathbf{S}$ *where* $\mathbf{F} =< xe^y, z - e^y, -xy >$ *where* S *is the ellipsoid* $x^2 + 2y^2 + 3z^2 = 4$.

(a) Fundamental Theorem of Line Integrals

(b) Green's Theorem

(c) Stokes' Theorem

(d) Gauss' Divergence Theorem

(e) Just calculate the surface integral directly.

Example 33.6. *Evaluate* $\int_c < e^y, xe^y + e^z, ye^z > \cdot d\mathbf{r}$ *where* c *is the line segment from* $(0, 2, 0)$ *to* $(4, 0, 3)$.

(a) Fundamental Theorem of Line Integrals

(b) Green's Theorem

(c) Stokes' Theorem

(d) Gauss' Divergence Theorem

(e) Just calculate the line integral directly.

Example 33.7. *Evaluate* $\iint_S curl\mathbf{F} \cdot d\mathbf{S}$ *for* $\mathbf{F} =< \arctan(x^2yz^2), x^2y, x^2z^2 >$ *and* S *is the cone* $x = \sqrt{y^2 + z^2}$, $0 \leq x \leq 2$ *oriented in the positive direction of the x-axis.*

(a) Fundamental Theorem of Line Integrals

(b) Green's Theorem

(c) Stokes' Theorem

(d) Gauss' Divergence Theorem

(e) Just calculate the surface integral directly.

Example 33.8. *Evaluate* $\int_c \mathbf{F} \cdot d\mathbf{r}$ *for* $\mathbf{F} =< 2y, xz, x + y >$ *and* c *is the intersection of the plane* $z = y + 2$ *and the cylinder* $x^2 + y^2 = 1$.

(a) Fundamental Theorem of Line Integrals

(b) Green's Theorem

(c) Stokes' Theorem

(d) Gauss' Divergence Theorem

(e) Just calculate the line integral directly.

33.8 Vector Calculus Flow Chart

{Vector Calculus}

$\int_C \vec{F}\cdot dr$

if $\vec{F}$ is conservative → find potential fn.

Fundamental Thm of Line Integrals

$= f(\vec{r}(b)) - f(\vec{r}(a))$

where $\vec{F} = \nabla f$

if is closed $= 0$

parameterize C using r(t) → $\int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

If $F=\langle P,Q,R\rangle$ Stokes

Stoke → **Stokes** $\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$

$\int_C P\,dx + Q\,dy$

if $\vec{F} = \langle P, Q\rangle$ use Green → Double Int $\iint_R \ldots\, dA$ ($Q_x - P_y$)

Green's Thm

→ $\iint \ldots\, dx\,dy$

→ $\iint \ldots\, dy\,dx$

→ $\iint \ldots\, r\,dr\,d\theta$

$\vec{n} = \langle -g_x, -g_y, 1\rangle$

or $\langle 1, -g_y, -g_z\rangle$ etc.

Calculate Surface Int. →

Surface Integrals

$\iint_S f(x, y, g(x,y))\,dS$

or $\iint_S f(x, g(x,z), z)\,dS$

or $\iint_S f(g(y,z), y, z)\,dS$

if S is explicitly given → $\iint_R \ldots\, dA$

if S is implicitly given

if S is parametric

use $dS = \sqrt{g_x^2 + g_y^2 + 1}\,dA$

Line Integrals:

$\int_C f\,ds$

$\int_C f\,dx$

$\int_C f\,dy$

$\int_C f\,dz$

$\oint f$ ← closed curve

parameterize C ↓

$\int_a^b f \ldots dt$

where $x = x(t)$ $dx = x'(t)\,dt$

$y = y(t)$ $dy = y'(t)\,dt$

$z = z(t)$ $dz = z'(t)\,dt$

$ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$

Surface Integral / Flux

$\iint_S \vec{F}\cdot\vec{n}\,dS$ or $\iint_{\partial S} \vec{F}\cdot d\vec{S}$

↓ Divergence thm

Triple Integral

$\iiint_S \text{div}(\vec{F})\,dV$

→ Cartesian Form: $\iiint \ldots\, dx\,dy\,dz$ or $\iiint \ldots\, dz\,dy\,dx$ ... etc

→ Cylindrical Form: $\iiint \ldots\, r\,dz\,dr\,d\theta$

→ Spherical Form: $\iiint \ldots\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

→ Change of variable: $\iiint \ldots\, |\text{Jacobian}|\,du\,dv\,dw$

A Practice Problems and Review for Exams

The following pages are practice problems and main concepts you should master on each exam.

These problems are meant to help you practice for the exam and are often harder than what you will see on the exam. You should also look over all ICE sheets, Homework problems, Quizzes, and Lecture Notes. Make sure you can do all of the problems in these sets.

Disclaimer: The following lists are topics that you should be familiar with, and these are problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

I will post the solutions to these practice exams on our blackboard site. If you find any mistakes with my solutions, please let me know right away and feel free to email at any hour.

Good Luck,
Dr. H

A.1 Exam 1 Review Problems

Review all examples from Notes and Ice sheets and quizzes. Know basic theorems and definitions. Here are some practice problems. Note you can use a calculator on all problems including sketching polar curves but you must show your work or explain your answer. Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Exam 1 will have 5 Mastery Concepts:

Concept 1: Parametric Equations, Vectors, and Lines in Space (Problems 1-4)

- Be able to differentiate parametric equations
- Be able to sketch parametric equations
- Be able to write equations of lines in space

Concept 2: Planes and Vector Valued Functions (Problems 5-10)

- Be able to find the unique equation of a plane that is perpendicular to a curve at a given point (use Tangent Vector)
- Be able to find the position, velocity, and acceleration functions and their relationships with each.
- Be able to show that a vector valued function lies on a given surface.

Concept 3: Curvature (Problem 11)

- You will be given the definition of curvature, but you will need to be able to compute it.

Concept 4: Surfaces (Problems 12,13,14)

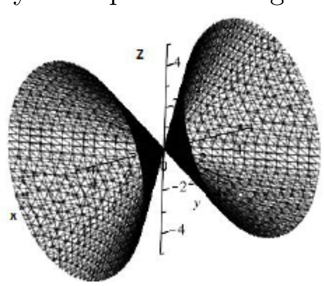
- Be able to recognize surfaces from an equation and vise versa.
- Be able to give verbal descriptions of traces of a surface in the planes $x = k, y = k$, and $z = k$.
- Be able to create and use a contour map and use it to recognize a surface.

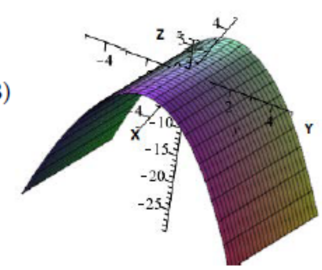
Concept 5: Polar Curves (Problems 16-18)

- Be able to convert between polar and Cartesian equations.
- Be able to sketch a polar curve
- Find the slope of the tangent line at a point on a polar curve.
- Find the area bounded by a polar curve.
- Be able to calculate derivatives of polar equations.

Practice Problems for Exam 1: Solutions are posted in blackboard.

1. Given the parametric curve $x = t^2 + 1$, $y = e^t - 1$.
   a) Find $\frac{dy}{dx}$
   b) Find $\frac{d^2y}{dx^2}$
2. You are at the point $(-1 - 3, -3)$ standing upright and facing the xz-plane. You walk 2 units forward and turn left and walk for another 2 unites. What is your final position?
3. Find parametric and symmetric equations of the line passing through the points $(-1, 0, 5)$ and $(4, -3, 3)$.
4. Find the equation of the tangent line for $\mathbf{r}(t) =< 12t^2, -4\cos t >$ at $t = 0$.
5. Find the equation of the plane perpendicular to the curve $\mathbf{r}(t) =< \cos(t), \sin(t), t >$ at $t = \dfrac{\pi}{2}$.
6. Find a vector orthogonal to both $< 1, -1, 0 >$ and $< 1, 2, 3 >$
7. Find an equation of a plane that passes through the point $(1, 3, 4)$ and is parallel to the plane $6x - 2y + 9z = 3$
8. Given any two vectors $\mathbf{u}$ and $\mathbf{v}$, explain why $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$.
9. Find the position and velocity vectors if $\mathbf{a} =< 12t^2, -4\cos t >$ and $\mathbf{v}(0) = -\mathbf{i}$ and $\mathbf{r}(0) = 2\mathbf{i} + \mathbf{j}$
10. Find the position function of a particle given that the velocity function is $\mathbf{v}(t) =< 2t, \cos(t) >$ with initial position $\mathbf{r} = \mathbf{i}$.
11. Find the curvature of the curve given by $x = t, y = \ln(t)$.
12. Identify the equation that generates the surfaces shown below.

A) 

B) 

$x^2 + y^2 + z^2 = 1$	$x = z^2 - y^2$	$z = 1 - x^2$	$z = 1 - y^2$
$x^2 + z^2 - x^2 = 1$	$x = z^2 + y^2$	$x = 1 - y^2$	$y = 1 - x^2$
$x^2 = y^2 + z^2$	$x^2 - y^2 - z^2 = 1$	$x = 1 - z^2$	$y = 1 - z^2$

13. Be able to recognize graphs of surfaces like $z = x^2 + y^2$, $z = x^2 - y^2$, $z = -x^2 - y^2$ etc.

14. Sketch the contour map for $f(x, y) = \sqrt{x + y}$. remember to label your contours!
15. Find the area inside one of the leaves in the curve $r = 4\cos(2\theta)$
16. Find the area inside both $r = 4 - 2\cos(\theta)$ and $r = 6\cos\theta$
17. Find the Cartesian coordinates of the point whose polar coordinates are $(-2, \frac{3\pi}{2})$
18. Find the slope of the tangent line in terms of θ for $r = 2 + 2\sin\theta$. For what θ is the tangent line horizontal?

A.2 Exam 2 Review Problems

Review all examples from Notes, Homework, Ice sheets, and Quizzes. Know basic theorems and definitions. Make sure you can do all of those problems. You can also practice with the problems below. Note you can use a calculator on all problems, but you must show your work. Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Exam 2 will have the concepts from Exam 1 along with the following additional Mastery Concepts:

Concept 6: Graphical and Conceptual Analysis of Partial Derivatives (Problems 1-5)

- Be able to determine the signs of higher partial derivatives given a surface
- Be able to determine the signs of higher partial derivatives given a contour map
- Be able to approximate partial derivatives given a table of values
- Be able to determine the signs of partial derivatives given an equation and be able to explain what this means for the rates of change of the problem.

Concept 7: Partial Derivatives (Directional Derivatives, Chain Rule, Gradient) (Problems 6-8, 10-15)

- Be able to use the Chain Rule to compute partial derivatives and implicit derivatives.
- Be able to compute directional derivatives.
- Be able to find the direction of the maximum and minimum rate of change

Concept 8: Limits (Problem 9)

- Be able to prove a limit does not exist by exhibiting two paths of approach with different limiting behavior.
- Be able to prove a limit exists by appealing to continuity or separating the variables.
- Be able to use Polar Coordinates to prove a limit exists or does not exist.

Concept 9: Optimization and Classifying Extrema (Problems 16-18)

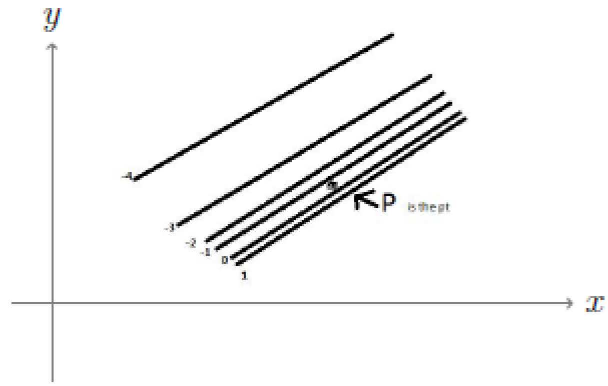
- Be able to find and classify critical points of a function.
- Be able to determine extreme values of a function on a closed, bounded domain.

Concept 10: Lagrange Multipliers (Problems 19-20)

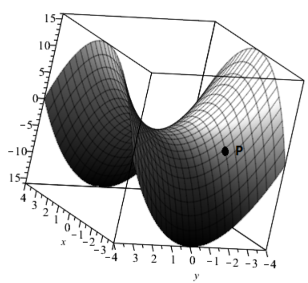
- Be able to use Lagrange Multipliers to find max/mins given a constraint.

Practice Problems for Exam 2: Solutions are posted in Blackboard.

1. Following the contour map for the function $f(x, y)$ determine the signs of $f_x(P)$, $f_y(P)$, $f_{xx}(P)$, $f_{yy}(P)$, and $f_{xy}(P)$.



2. Consider the surface below which represents $f(x, y)$ Determine the signs of $f_{xx}(P)$ and $f_{yy}(P)$



3. Suppose that the price P (in dollars) to purchase a used car is a function of its original cost C (in dollars) and its age A (in years). A possible model for this function is $P = Ce^{0.02A}$. What is the sign of $\dfrac{\partial P}{\partial C}$? Explain in one sentence what your previous answer above means in the context of the cost of the used car.

4. The following table gives values of the function $f(x, y)$. (The row headings are x-values and the column headings are y values.) Use the table to approximate $f_x(5, 4)$ and $f_y(5, 4)$

x // y	**0**	**2**	**4**	**6**	**8**
0	15	17	18	19	20
5	21	23	25	26	28
10	35	37	39	40	42
15	57	59	60	62	65

5. Let $f(z, y) = \ln(x - y)$. Find the mixed partial derivatives of f_{xy} and f_{yx}

6. The part of a tree normally sawed into lumber is the trunk, a solid shaped approximately like a right circular cylinder. If the radius of the trunk of a certain tree is growing 1/2 in/year and the height is increasing 8in/year, how fast is the volume increasing when the radius is 20 inches and the height is 400 inches?

7. Sand is pouring onto a conical pile in such a way that at a certain instant the height is 100 inches and increasing at 3 in/min and the base radius is 40 inches and increasing at 2 in/minute. How fast is the volume increasing at that instant?

8. Find $\dfrac{\partial z}{\partial t}$ if $z = x^2 y, x = 2s + t$, and $y = 1 - st^2$.

9. Evaluate the following limits or explain why it do not exist. Justify your answer:

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|xy|}{xy}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + 2y^2}$

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y(e^x - 1)}{x}$

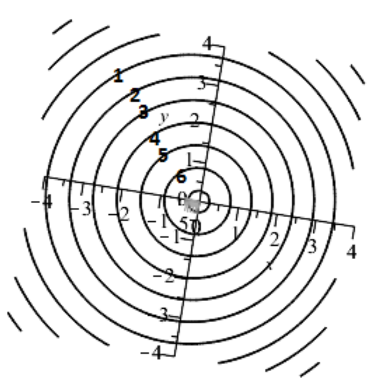
   (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{yx}{\cos(x)}$

10. Find the directional derivative of $h(x, y, z) = xyz$ at the point $P = (2, 1, 1)$ in the direction of $\mathbf{v} =< 2, 1, 2 >$.

11. What is the direction of the maximum increase of $f(x, y, z) = xe^y$ at the point $P = (2, 0, 2)$?

12. Let $f(x, y, z) = x^3 + 3xz + 2yz + z^2$. Find the greatest rate of increase of f at the point $(1, -2, 1)$ and a unit vector pointing in the direction of greatest increase.

13. The figure below is a contour map of a function f(x, y). At the point $(2, 2)$, sketch a unit vector in the direction of $\nabla f(2, 2)$.



14. Find the equation of the tangent plane to $z = xe^{-2y}$ at (1,0,1).

15. Approximate Δz for $z = \ln(xy^2)$ as (x, y) moves from $(4, -2)$ to $(2, -1)$ using the differential.

16. Find the largest and smallest value of $f(x, y) = x - x^2 - y^2$ if x and y are on the unit circle.

17. Find the critical points of $f(x, y) = x^4 + y^4 - 4x - 32y + 10$. Classify them using the Second derivative test.

18. Find the critical points of $f(x, y) = xye^{-x-y}$. Classify them using the Second derivative test.

19. Use Lagrange Multipliers to find the extreme values of$f(x, y) = xy$ subject to the constraint $4x^2 + y^2 = 8$.

20. Use Lagrange Multipliers to find the extreme values of $f(x, y) = xe^y$ subject to the constraint $x^2 + y^2 = 2$.

A.3 Exam 3 Review Problems

Review all examples from Notes, Homework, Ice sheets, and Quizzes. Know basic theorems and definitions. Make sure you can do all of those problems. You can also practice with the problems below. Note you can use a calculator on all problems, but you must show your work. Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Exam 3 will have the concepts from Exam 1-2 along with the following additional Mastery Concepts:

Concept 11: Double Integrals (See problems 1-6)

- Be able to set up and compute double integrals using both Cartesian and Polar Coordinates for general regions.
- Be able to switch the order of integration for a double integral.

Concept 12: Triple Integrals (See problems 7-11)

- Be able to set up and compute triple integrals using Cartesian, Cylindrical, and Spherical Coordinates depending on the regions of integration.
- Be able to switch the order of integration for a triple integral.

Concept 13: Change of Variables
(See problems 12,13, and notes, ICE, and HW examples)

- Be able to set up an integral by using change of variables. You will not need to compute the integral, but you will need to set up the integral completely using the new variables.

Concept 14: The Fundamental Theorem of Line Integrals (See problems 14-19)

- Be able to prove a Vector Field is conservative.
- Be able to find a potential function.
- Be able to use the potential function to calculate a line integral using the fundamental theorem of line integrals.

Concept 15: Vector Fields (See problems 20-21, notes)

- Know how to graph and recognize graphs of vector fields.
- Be able to tell whether or not a vector field is conservative/path independent/gradient field
- Look at Vector Field and Curl and Divergence Notes and HW

Practice Problems for Exam 3: Solutions are posted in Blackboard.

1. Evaluate $\iint_R \dfrac{1}{\sqrt{16-x^2-y^2}}dA$ where $R=\{(x,y): x^2+y^2\le 4, x\ge 0, y\ge 0\}$

2. Consider a plane given by the equation $z=ax+by+c$, where $a,b,c>0$. Find the volume of the solid lying under the plane and above the rectangle $R=\{(x,y)|0\le x\le 1, 0\le y\le 1\}$. Your answer may depend upon a,b,c.

3. Set up and evaluate an iterated integral that represents the surface area of $f(x,y)=2x+\frac{2}{3}y^{\frac{3}{2}}$ over the triangle with vertices $(0,0), (2,0)$, and $(2,2)$.

4. Change the order or integration for $\int_0^2\int_{x^2}^{2x} f(x,y)dydx$

5. Sketch the following region and write an iterated integral of a continuous function f over $R=\{(x,y): 0\le x\le 4, x^2\le y\le 16\}$

6. a) Evaluate $\iint_R xydA$ where R is bounded by $x=0, y=2x+1$, and $y=-2x+5$.

   b) Then change the order of the limits of integration.

7. $\iiint_V (1)dV$ where V is the region in the first octant bounded by $y=2x^2$ and $y+4z=8$.

8. Evaluate $\int_{-1}^1\int_{3x^2}^{4-x^2}\int_0^{6-z} dydzdx$

9. Use cylindrical coordinates to find the volume of the solid bounded above by the sphere centered at the origin having radius 5 and below by the plane $z=4$.

10. Find the volume of the region above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.

11. Swap the limits of integration for $\int_0^2\int_0^{4-2y}\int_0^{4-2y-z} dxdzdy$ to $dzdydx$

12. Find the Jacobian for the transformation $x=r\sin t$ and $y=r\cos t$.

13. Use a change of variables to rewrite the integral (you do not have to evaluate the integral). $\iint_R \sqrt{\dfrac{x-y}{x+y+1}}dA$ where R is the square with vertices $(0,0),(1,-1),(2,0),(1,1)$. Sketch the original region and the new region.

14. Consider the vector field: $\mathbf{F}(x,y,z)=yze^{xz}\mathbf{i}+e^{xz}\mathbf{j}+xye^{xz}\mathbf{k}$
    a) Determine Div$\mathbf{F}$
    b) Determine Curl$\mathbf{F}$
    c) Determine whether $\mathbf{F}$ is conservative
    d) Find a potential function for $\mathbf{F}$

15. Determine $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is a simple curve from (0,1,0) to (1,1,1) and $\mathbf{F}=\nabla(3xyz+xe^z)$

16. If $\mathbf{F} = (x^2 - y^2)\mathbf{i} + 2xyz\mathbf{j} + z^2\mathbf{k}$
    a) Find div$\mathbf{F}$

    b) Find curl$\mathbf{F}$

    c) Can we use the fundamental theorem of line integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for simple curve C?

17. Confirm that $\mathbf{F}(x, y) =< 2e^y - ye^x, 2xe^y - e^x >$ is conservative and find a potential function. Then what is the value of $\int_c \mathbf{F} \cdot d\mathbf{r}$ where C is a simple curve from (1,1) to (0,1).

18. Find a function f satisfying $\nabla f = (2xy + y)\mathbf{i} + (x^2 + x + \cos y)\mathbf{j}$

19. Find a function f satisfying $\nabla f = (yz - e^{-x})\mathbf{i} + (xz + e^y)\mathbf{j} + xy\mathbf{k}$

20. Know how to graph and recognize graphs of vector fields.

21. Be able to tell from the graph or equation whether or not a vector field is conservative/path independent/gradient field

A.4 Exam 4 Review Problems

Review all examples from Notes and Ice sheets and quizzes. Know basic theorems and definitions. Here are some practice problems. Note you can use a calculator on all problems including sketching polar curves. Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course. Review all examples from Notes, Homework, Ice sheets, and Quizzes. Know basic theorems and definitions. Make sure you can do all of those problems. You can also practice with the problems below. Note you can use a calculator on all problems. Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Exam 4 will have the concepts from Exam 1-3 along with the following additional Mastery Concepts:

Note: I have not identified specific problems since part of the the assessment is recognizing which technique to use!

Concept 16: Line Integrals and Green's Theorem

- Be able to recognize when we can use Greens Theorem and use it appropriately
- Be able to parameterize a curve so we can calculate the line integral without using the fundamental theorem of line integrals.

Concept 17: Surface Integrals

- Be able to set up and compute surface integrals and flux integrals.

Concept 18: Stokes and Gauss Divergence Theorems

- Be able to use Stokes and Divergence Theorems to set up and compute integrals.
- Be able to rewrite integrals using Stokes and Divergence Theorems.

Practice Problems for Exam 4: Solutions will be posted in Blackboard.

1. Evaluate $\int_C xyds$ where C is the circle with radius 2 on the oriented path from $(2,0)$ to $(0,2)$.
2. Evaluate $\int_C xyds$ where C is the circle with radius 2 on the oriented path from $(0,2)$ to $(2,0)$.
3. What is the relationship between $\int_C fds$ and $\int_{-C} fds$?
4. Evaluate $\int_C xyds$ where C is the line from $(2,0)$ to $(0,2)$.
5. Evaluate $\int_C xyds$ where C is the line from $(0,2)$ to $(2,0)$.
6. Compute $\int_C (x^2-y)dx + (y^2+x)dy$ from $(0,1)$ to $(1,2)$ along the curve $C: y = x^2+1$
7. A particle travels along the helix C given by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$ and is subject to the force $\mathbf{F}(x,y,z) =< x, z, xy >$. Find the total work done on the particle by the force for $0 \le t \le 3\pi$.
8. Evaluate $\int_C \sqrt{1+x^3}dx + xydy$ where C is a positively oriented triangle with vertices (0,0), (2,0), and (0,3).
9. Use Green's theorem to evaluate $\oint_C (3x-4y)dx + (4x-2y)dy$ where C is the path counterclockwise around the ellipse $x^2+4y^2 = 16$ beginning and ending at $(4,0)$
10. Use Green's theorem to evaluate $\oint_C (x^2y)dx + (2x)dy$ where C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$ oriented clockwise.
11. Verify that the line integral and surface integral of Stokes' Theorem are equal for the following vector fields:
    a) $\mathbf{F} =< 0, -x, y >$ where S is the upper half of the sphere $x^2+y^2+z^2 = 4$ and C is the circle $x^2+y^2 = 4$ in the xy plane.
    b)$\mathbf{F} =< x, y, z >$ where S is the paraboloid $z = 8-x^2-y^2$ for $0 \le z \le 8$ and C is the circle $x^2+y^2 = 8$ in the xy plane.
12. Use Stokes' Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F} =< -y, -x-z, y-x >$ where C is the boundary of the part of the plane $z = 6-y$ that lies in the cylinder $x^2+y^2 = 16$.
13. Calculate $\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}dS$ where $\mathbf{F} =< x+z^2, y-z^2, x >$ where S is the solid $0 \le y^2+z^2 \le 1$, $0 \le x \le 2$.
14. Calculate $\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}dS$ where $\mathbf{F} =< x^2, y^2, z^2 >$ where S is the solid enclosed by $x+y+z = 4$, $x = 0, y = 0, z = 0$.
15. Evaluate the flux of $\mathbf{F} =< x, y, 3 >$ across the sphere $x^2+y^2+z^2 = 1$.
16. Evaluate $\iint_G (2y^2+z)dS$ where $G: z = x^2-y^2$ for $0 \le x^2+y^2 \le 1$.
17. Evaluate $\iint_G ydS$ where $G: z = 4-y^2$ for $0 \le x \le 3$, $0 \le y \le 2$.
18. Find the work done by $\mathbf{F} = y^2\mathbf{i} + 2xy\mathbf{j}$ in moving an object from $(1,1)$ to $(3,4)$
19. Evaluate $\oint_C (e^{3x}+2y)dx + (x^2+\sin y)dy$ where C is the rectangle with vertices $(2,1), (6,1), (6,4), (2,4)$

B Homework

Please write your solutions on these homework pages and show enough of your work so that I can follow your thought process. This makes it easier for me to grade. Also please **staple** the homework together before you turn it in. Sometimes I have my stapler, but there is also a stapler in my office and at the front desk of the Department of Computer and Mathematical Sciences.

Follow the instructions for each question. If I can't read your work or answer, you will receive little or no credit!

B.1 Calculus III HW 1: Due Fri 9/1 Name: ____________

There are 3 sides! Do not use a calculator on this homework. Do not use Tables! If you do not show sufficient work, you may not receive full credit.

1. Evaluate $\int_0^8 xe^x dx$

2. Evaluate $\int \sqrt{3-x^2}dx$

3. Evaluate $\int \frac{x^2}{4 - x^2} dx$

4. Evaluate $\int \sin^3 x \sqrt{\cos x} dx$

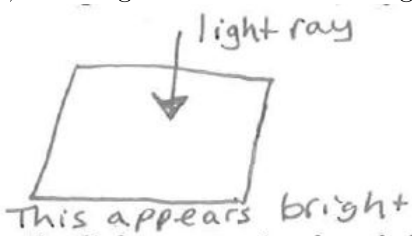
5. Find the equation(s) of the tangent line(s) to $x = \sin t$, $y = \sin(t + \sin(t))$ at $(0, 0)$. for $0 < t < 2\pi$.

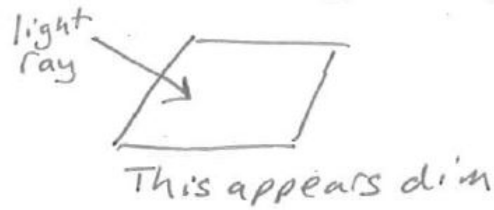
6. Determine $\dfrac{d^2y}{dx^2}$ without eliminating the parameter for $x = 3 - 2\cos t$, $y = -1 + 5\sin t$ for $t \neq n\pi$ for natural number n.

B.2 Calculus III HW 2: Due Fri 9/8 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers.

1. One way to accurately render three-dimensional objects on a computer screen involves using the dot and cross products. In order to determine how to shade a piece of a surface, we need to determine the angle at which rays from the light source hit the surface. If the light ray hits the surface straight on, then this piece of the surface will appear bright. On the other hand, if the light comes in on an angle, this piece of the surface will not appear as bright.



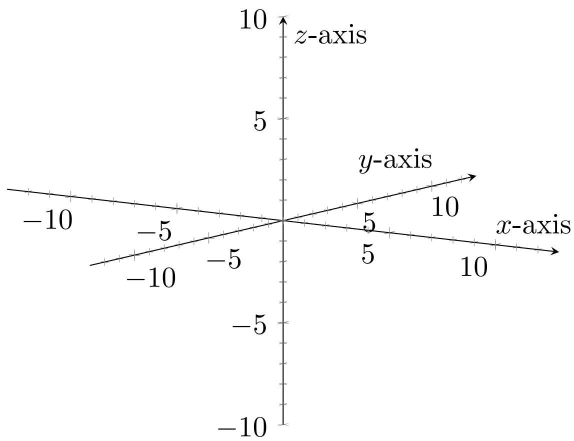


Suppose the light source is placed above the xy-plane, but at an angle, so the light rays come in parallel to the vector $< 0, 1, -1 >$.

a) Consider the plane containing the points (3,2,4), (2,5,3), and (1,2,6). Find two vectors that are in the plane, and then determine a vector orthogonal (normal) to the plane.

b) Find the angle between the light rays an the normal vector to the plane.

2. You are at the point $(-1,-3,-3)$ standing upright and facing the yz plane. You walk 2 units forward, turn left, and walk for another 2 units.
a) What is your final position?
b) From the point of view from your position (and where you are facing in the coordinate system), are you above, below, in front of, behind, to the right of, or to the left of the yz-plane?
c) Are you above, below, to the right of, or to the left of the xy-plane?



3. A woman walks due west on the deck of a ship at 3 mph. The ship is moving north at a speed of 22 mph. Find the speed of the woman relative to the surface of the water.

4. The Amazing Math Kitties, Eva and Archer passed Calculus II and are now in Calculus III. Today they are working on equations in 3-space. Eva says the graph is a plane, but Archer says the graph of the equation $y = 3x + 2$ is a line that has a slope of 3 and a y intercept of 2. Who is correct? Are they both correct? Explain.

5. A wagon is pulled a distance of 100meters along a horizontal path by a constant force of 50N. The handle of the wagon is held at an angle of $30°$ above the horizontal. How much work is done?

6. A 100-meter dash is run on a track in the direction of the vector $\vec{v} = 4\vec{i} + 3\vec{j}$. The wind velocity $\vec{w}$ is $6\vec{i} + 4\vec{j}$ km/hr. The rules say that a legal wind speed measured in the direction of the dash must not exceed 5 km/hr. Will the race results be disqualified due to an illegal wind? [Hint: think projections...]

7. Now the fabulous Math Kitties are talking about vectors. Archer says that he thinks $|\mathbf{u}+\mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$. Is Archer's statement true? Explain.

8. Please review conic sections by either reading the posted notes in Blackboard or by reading section 10.5 in your textbook.

   Check one:
   ☐ I have reviewed conic sections.
   ☐ I have not reviewed conic sections.

   More problems on next page (or back)

9. Eva and Archer, the amazing math kitties, are working on their homework. Unfortunately, Archer spilled milk on the one of the questions so all he can see is

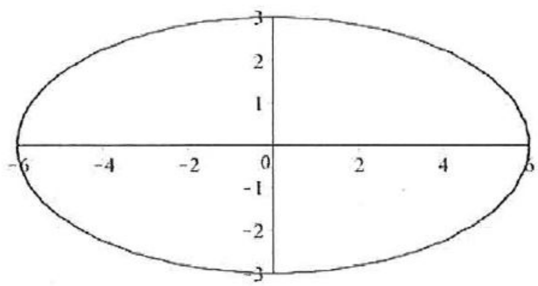
$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \underline{\hspace{3cm}}.$$

Eva tells him he can still answer the question. Why is Eva correct and what is the answer to the question?

B.3 Calculus III HW 3: Due Fri 9/15 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers.

1. Consider the ellipse given by the vector function $\mathbf{r}(t) = 6\cos t\mathbf{i} + 3\sin t\mathbf{j}$ and its graph below:



   (a) Sketch the vectors $\mathbf{T}(\frac{\pi}{2})$ (the unit tangent vector) and $\mathbf{N}(\frac{\pi}{2})$ (the principle unit normal vector) at the point on the ellipse corresponding to $t = \frac{\pi}{2}$. Make sure to denote which is which. (You should not need to do any calculations.)

   (b) By looking at the graph, decide at which two points on the ellipse the curvature is minimal. List the two points.

2. Archer wants to find the position function of a particle given that the velocity function is $v(t) = 2t\mathbf{j} + \sin(t)\mathbf{k}$ and the initial position is $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$. Archer says this means the constant of integration is $< 1, 1, 0 >$, but Eva warns him to be careful about using the initial position properly. Help Archer find the correct position function.

3. Find the symmetric equations of the line passing through the points $(-1, 0, 5)$ and $(4, -3, 3)$.

4. Given the following curve: $\mathbf{r}(t) =< 2t, 4\sin(t), 4\cos(t) >$, answer the following :
   a) Find the unit tangent vector.

   b) Find the curvature κ of $\mathbf{r}(t)$.

5. The exquisite math cats, Eva and Archer, are trying to determine the tangential and normal components of acceleration (a_T and a_N) for $\mathbf{r}(t) =< t^3, t^2 >$. Archer says to find a_T, he should dot $< 3t^2, 2t >$ with $< 6t, 2 >$ Eva says that he is close but not quite right. What did Archer forget? Then find a_T and a_N.

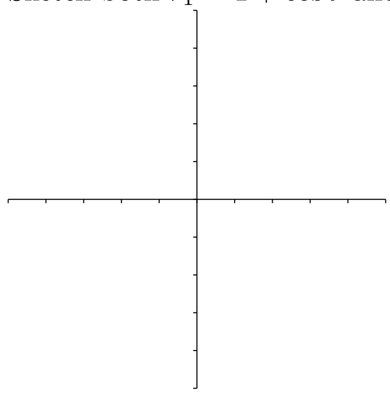
6. Professor Keleher is traveling along the curve given by $\mathbf{r}(t) =< -2e^{3t}, 5\cos t, -3\sin(2t) >$. If the power thrusters are turned off, his ship flies off on a tangent line to $\mathbf{r}(t)$. He is almost out of power when he notices that a station on Octapa is open at the point with coordinates $(1.5, 5, 3.5)$. Quickly calculating his position, he turns off the thrusters at $t = 0$. Does he make it to the Octapa station? Show your work, and explain your answer in one complete sentence.

B.4 Calculus III HW 4: Due Fri 9/22 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers.

1. Remember, you can use a calculator to help you.

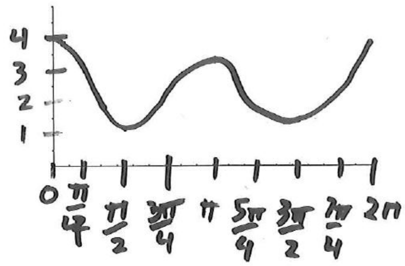
   (a) Sketch both $r_1 = 1 + \cos\theta$ and $r_2 = 1 - \cos\theta$ on the same axis. **Label your curves.**

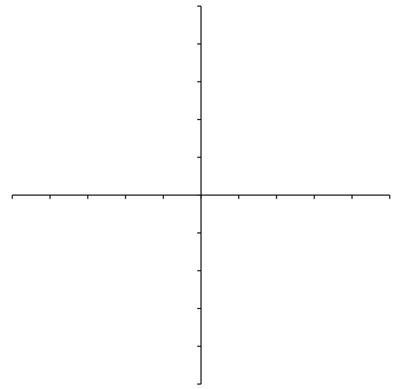


   (b) Find the area of the region that lies inside both $r_1 = 1 + \cos\theta$ and $r_2 = 1 - \cos\theta$. Even if you use a calculator, you must show work and the integral you set up.

   (c) Find the area of the region that lies inside $r_1 = 1 + \cos\theta$ but not $r_2 = 1 - \cos\theta$. Even if you use a calculator, you must show work and the integral you set up.

2. Below is the Cartesian graph of $y = f(x)$ with $0 \leq x \leq 2\pi$. Using the same function f, sketch a graph in polar coordinates for $r = f(\theta)$.

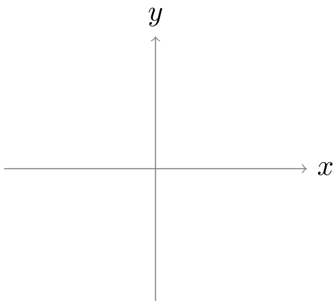




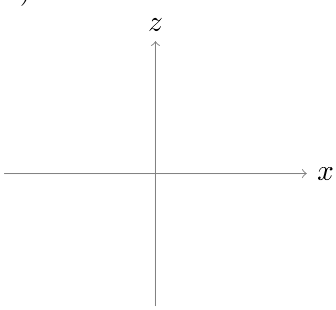
3. Find the slope of the tangent line to $r = \cos(2\theta)$ at $\theta = \frac{\pi}{4}$.

4. Consider the surface $\dfrac{y^2}{9} + \dfrac{z^2}{36} - \dfrac{x^2}{4} = 1$.

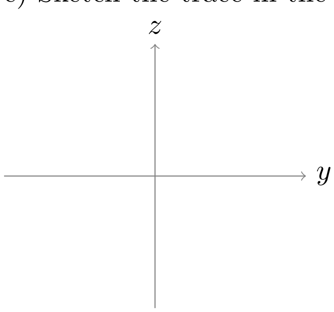
a) Sketch the trace in the xy plane.



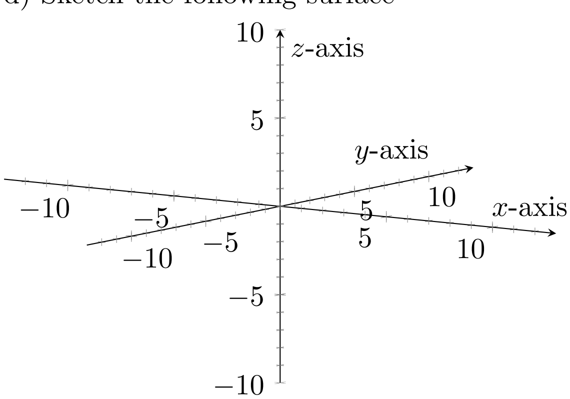
b)Sketch the trace in the xz plane.



c) Sketch the trace in the yz plane.



d) Sketch the following surface



B.5 Calculus III HW 5: Due Fri 9/29 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers.

1. The Amazing Math Kitties are working on evaluating limits in Calculus 3. First they are exploring $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xe^y - x}{y}$.

   (a) Archer says that because he gets $\frac{0}{0}$ when he plugs in 0 for x and y, the limit is undefined. Eva reminds him sometimes we still have limits exist with this indeterminate form. In fact we could use L'Hopital's Rule to show $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x} = 1$. Use L'Hopital's Rule to verify Eva's example.

   (b) Archer, now convinced, excitedly says that he will now use L'Hopital's Rule for this limit. Eva says first he needs to separate the limit before he can use L'Hopital. Help Archer separate the limit into the product of two singular variable functions. Then use L'Hopital's Rule to help Archer calculate the limit.

2. Now Eva and Archer are working on $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2 - y^2}}$:

   (a) Archer says that he will use L'Hopital's Rule for this limit since it worked in the last example. Eva shakes her head. Why does Eva not agree? Why does this method not work in this example?

(b) Archer says he will now try to evaluate this limit by trying the path $x = 0$, $y = y$. "Careful!" Eva warns. "That is not a valid path we can use." Explain why this is not a valid path.

(c) Help Archer find at least two different paths that go to two different limits. What does this mean about our limit?

(d) Eva says that she can use Polar Coordinates to solve $\lim\limits_{(x,y)\to(0,0)} \dfrac{y}{\sqrt{x^2 - y^2}}$. Use Eva's method to show this limit doesn't exist.

3. Consider $f(x, y, z) = y^2 \sin(x^3 + z^2)$

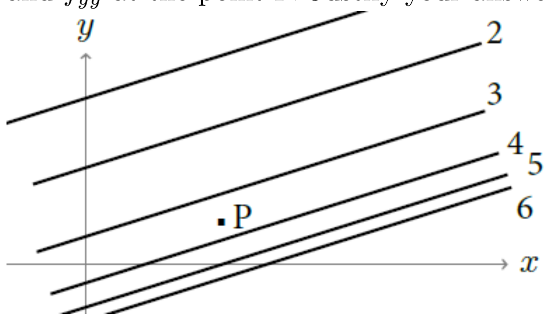
(a) Determine f_x and f_z.

(b) Determine f_{xx} and f_{zx}.

(c) Archer is attempting to determine f_{zxxyyy}. Eva says that he can easily justify that the answer is 0 without doing any computation. Help Eva explain why this is so.

4. Suppose that the price P (in dollars), to purchase a used car is a function of C, its original cost (also in dollars), and its age A (in years). So P = f(C,A). Is the sign of $\dfrac{\partial P}{\partial C}$ positive, negative , or zero. Explain.

5. The following is a contour map for the function $f(x, y)$. Determine the signs of f_x, f_y, f_{xx}, and f_{yy} at the point P. Justify your answers.



B.6 Calculus III HW 6: Due Fri 10/13 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers.

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. Designing safe boilers depends on knowing how steam behaves under certain changes in temperature and pressure. Steam tables, such as the one below, are published giving values of the function $V = f(T, P)$ where V is the volume (in cubic feet) of one pound of steam at a temperature of T (in degrees Fahrenheit) and pressure P (in pounds per square inch).

T// P	**20**	**22**	**24**	**26**
480	27.85	25.31	23.19	21.39
500	28.46	25.86	23.69	21.86
520	29.06	26.41	24.20	22.33
540	29.66	26.95	24.70	22.79

a) Find the tangent plane to $V = f(T, P)$ for T near 520°F and P near 24 lb/in^2. [Hint: Use the tables to approximate your partial derivatives and recall the equation of our tangent plane is: $V = V_T(T - T_0) + V_P(P - P_0) + f(T, P)$]

b) Use the tangent plane to estimate the volume of a pound of steam at a temperature of 525°F and P near 24.3 lb/in^2.

2. Find the directional derivative of $f(x, y) = \ln(xy)$ at $(\frac{1}{2}, \frac{1}{4})$ in the direction of $< 1, 1 >$.

3. a) Find the maximum and minimum rates of change of $f(x, y, z) = e^{xyz+1}$ at $(0, -1, 1)$ and the directions in which they occur. Write your answer in the forms below.

We have a maximum rate of change of __________ in the direction of __________

We have a minimum rate of change of __________ in the direction of __________

b) Find the unit vector in the direction of maximum increase of f at $(0, -1, 1)$.

4. The Super Math Kitties, Eva and Archer are exploring the gas equation for one mole of oxygen relates its pressure P (in atmospheres), its temperature, T (in K), and its volume, V (in cubic decimeters):
$T = 16.574\frac{1}{V} - 0.52754\frac{1}{V^2} + 0.3879P + 12.187VP$.

a) What does T_V represent? Archer say that the T_V is the rate at which the temperature changes when pressure increases and the volume is kept constant. Is he right? If not, adjust his representation.

b) Help Archer find T_V when the volume is 25 dm^3 and the pressure is 1 atmosphere.

c) Now determine T_P when the volume is 25 dm^3 and the pressure is 1 atmosphere.

d) What does the total differential of T, dT, represent?

e) Find an expression for the **total differential** dT in terms of the differentials dV and dP when the volume is 25 dm^3 and the pressure is 1 atmosphere. Archer gives you a hint that $dT = < T_V, T_P > \cdot < dV, dP >$ and Eva reminds you that your answer should have the terms dT, dV, and dP in it and to use your answers from b and c.]

f) Eva now wants to use your answer above to estimate how much the volume would have to change if the pressure increased by 0.1 atmosphere and the temperature remained constant. Archer reminds you that we are approximating ΔV using dV.

g) Help Archer write your answer in the previous question in the form of "the volume is __________________ at a rate of _________ cubic decimeters".

B.7 Calculus III HW 7: Due Fro 10/20 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers.

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

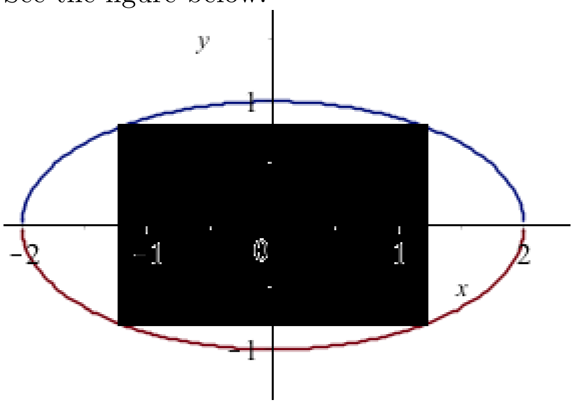
1. Given $3x^2z + y^3 - xy^2z^3 = 0$, use Calculus III techniques to determine the following:

   a) $\dfrac{\partial y}{\partial x}$

   b) $\dfrac{\partial z}{\partial x}$

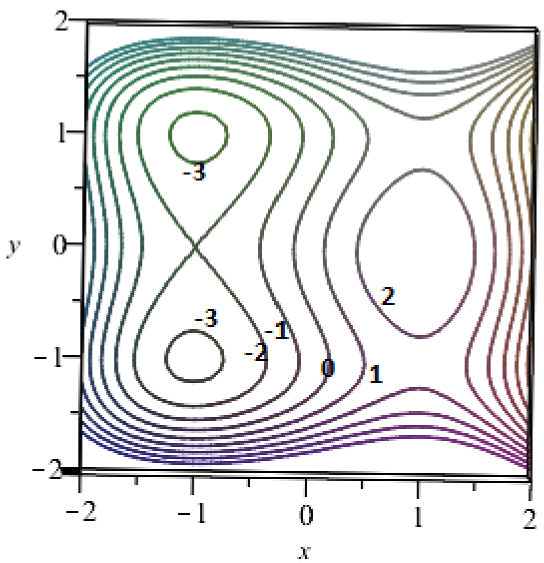
2. Wheat production in a given year, W, depends on the average temperature T and the annual rainfall R. Scientists estimate that the average temperature is rising at a rate of 0.14°/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels, $\dfrac{\partial W}{\partial T} = -2$ and $\dfrac{\partial W}{\partial R} = 5$. Estimate the current rate of change of wheat production $\dfrac{dW}{dt}$.

3. The crafty math kitties, Eva and Archer, want to cut a rectangular beam with maximal rectangular cross section from an elliptical shaped log with semi-axes of lengths 1 foot and 2 feet. Help the kitties use find the maximal cross-sectional area of such a beam using Lagrange multipliers. Include correct units in your answer and show your work. (Hint: You need to create the cross section function that you want to maximize. And if coordinate axes are set up so the center of the log is at the origin, the log is then bounded by the ellipse $x^2 + 4y^2 = 4$. See the figure below.



4. Find and classify all six critical points of $f(x,y) = 3x - x^3 - 2y^2 + y^4$.
Write your answers in the chart below

Critical Point	Value of **D**	sign of f_{xx}	**Conclusion**



B.8 Calculus III HW 8: Due Fri 11/3 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers.

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. Eva and Archer want to calculate $\int_{-2}^{2}\int_{0}^{2} |3x^3y^2|dydx$.

   a) Archer is a little confused by the absolute values, he uses his absolute value properties to rewrite $|3x^3y^2|$ as $3|x^3|\cdot|y^2|$. Eva says that this is an excellent start since it allows them to separate the integral into $\int_{-2}^{2}\int_{0}^{2} |3x^3y^2|dydx = \int_{-2}^{2} |x^3|dx \cdot \int_{0}^{2} 3|y^2|dy$. Is Eva correct? Why or why not?

   b) Archer is still a little stuck, but Eva says he can use symmetry to help eliminate the absolute values. In particular, help Archer rewrite $\int_{-2}^{2} |x^3|dx$ without absolute values. Then evaluate the integral.

c) Now help Archer rewrite $\int_0^2 |3y^2|dy$ without absolute values. Then evaluate the integral.

d) Use your answers from part b and c to calculate $\int_{-2}^2 \int_0^2 |3x^3y^2|dydx$.

2. Set up, but **do not** evaluate an integral that represents the volume of a tetrahedron bounded by the coordinate planes and the plane $z = 6 - 2x - 3y$.

3. a)Set up the integral $\iint_S (x^2 - xy)dA$, where S is the region between $y = x$ and $y = x^2$ using $dydx$.

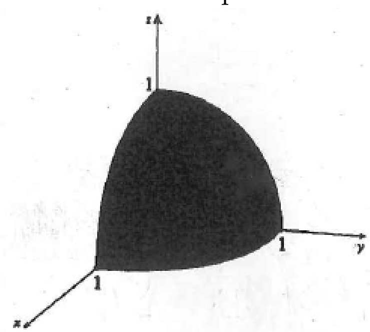
b) Set up the integral using $dxdy$

4. Find the coordinates of the center of mass of the following plane region: the quarter disk in the first quadrant bounded by $x^2 + y^2 = 4$ with density $\delta(x, y) = 1 + x^2 + y^2$. Recall the center of mass is given by $(\bar{x}, \bar{y})$ where $\bar{x} = \dfrac{\iint_R x\delta(x, y)dA}{\iint_R \delta(x, y)dA}$ and $\bar{y} = \dfrac{\iint_R y\delta(x, y)dA}{\iint_R \delta(x, y)dA}$. Archer says to use symmetry!

B.9 Calculus III HW 9: Due Fri 11/10 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. Consider the double integral $\iint_S (x^2 + y^2 + z^2)dV$ where S is the following solid, which is the piece of the unit sphere in the first octant.

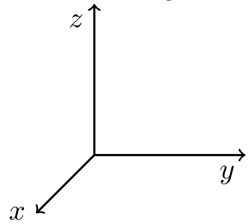


   (a) Set up the triple integral as an iterated integral in rectangular coordinates. DO NOT EVALUATE!

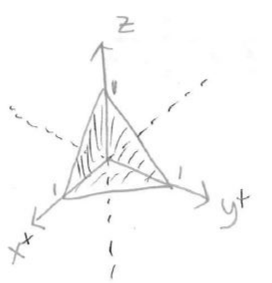
   (b) Set up the triple integral as an iterated integral in cylindrical coordinates. DO NOT EVALUATE!

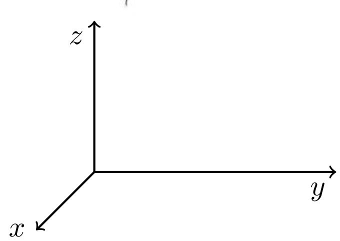
   (c) Set up the triple integral as an iterated integral in spherical coordinates. DO NOT EVALUATE!

2. Change the order of the integration for $\int_0^2 \int_0^{4-2y} \int_0^{4-2y-z} f(x,y,z)dxdzdy$ to $dzdydx$. AND sketch the region of the integration.



3. Archer and Eva are trying to sketch the solid region of integration for the triple integral $I = \int_0^1 \int_0^1 \int_0^{1-y} 2x dx dy dz$.
Archer has graphed the region below. Is he correct? If not, fix his graph. Eva says it helps her to draw the footprints in each of the coordinate planes (xy, xz, and yz.)





4. a) Set up a **double** integral in polar coordinates that represents the volume of the solid region enclosed by the paraboloid $z = x^2 + y^2$ and the plane $r = 4$.

b) Set up a **triple** integral that represents the volume of the solid region enclosed by the paraboloid $z = x^2 + y^2$ and the plane $r = 4$.

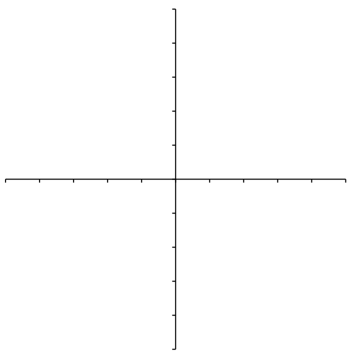
5. **Evaluate** the integral $\iiint_D \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} dV$ where D is the region between the spheres of radius 1 and 2 centered at the origin. Eva says to use a good coordinate transformation!

B.10 Calculus III HW 10: Due Fri 11/17 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. a) Sketch each vector field below and use the sketch to decide whether the divergence is positive, negative, or zero at the point $(1, 1)$. Recall, the divergence at a point calculates the net outflow per unit of volume. The divergence is positive if the net outflow is positive. It is negative if the net outflow is negative (we have inflow). When the divergence is zero, the net outflow is zero and we say our vector field is incompressible.
   b) Check your answer by calculating the divergence of $\mathbf{F}$.
   c) Determine whether the vector field is conservative or not.

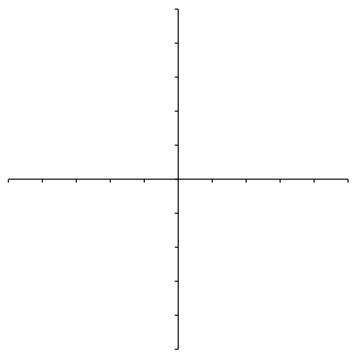
a) $\mathbf{F}(x, y) = -y\mathbf{j}$



Inflow, Outflow, or no flow:
Divergence =

Conservative?

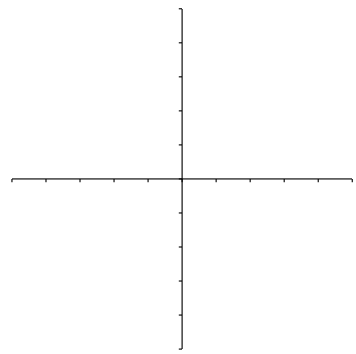
b) $\mathbf{F}(x, y) = < y, y + 2 >$



Inflow, Outflow, or no flow:
Divergence =

Conservative?

c) $\mathbf{F}(x, y) = \mathbf{j} - \mathbf{i}$

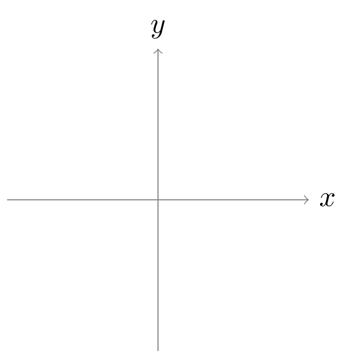


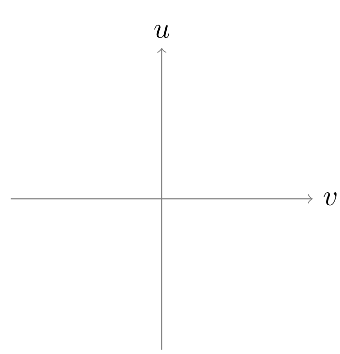
Inflow, Outflow, or no flow:
Divergence =

Conservative?

2. a) Evaluate $\iint_R \sqrt{y^2 - x^2}\,dA$, where R is the diamond bounded by $y - x = 0, y - x = 2, y + x = 0$, and $y + x = 2$. Eva notices that the bounds can be used to guide us to a nice u and v. Archer agrees and says that letting $u = y - x$ will also help with the integrand since $y^2 - x^2 = (y - x)(y + x)$.

b) Sketch the original and new regions of integration.





3. Consider the vector field, $\mathbf{F}(x, y, z) =< yz, xz, (xy + 2z) >$

a) Calculate the curl of $\mathbf{F}$ and show that $\mathbf{F}$ is a conservative vector field.

b) Find a potential function f for $\mathbf{F}$.

c) Use your answer from part b and the Fundamental Theorem of Line Integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

B.11 Calculus III HW 11: Due Mon 12/2 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. Water is an essentially incompressible fluid, that is the divergence of a velocity field representing the flow of water is 0. Determine which of the following vector fields could represent the velocity vector field for water flowing.
   a) $\mathbf{F}(x, y, z) = xy\mathbf{i} + xz\mathbf{j} - yz\mathbf{k}$

   b) $\mathbf{F}(x, y, z) = < xy, z\sin x, e^{xy} >$

2. Evaluate $\int_C (2x - 3y)ds$ where C is the line segment from $(-1, 0)$ to $(0, 1)$.

3. Evaluate $\iint_G xy dS$ if G is the plane $z = 2 - x - y$ in the first octant.

4. Suppose we have the force field $\mathbf{F} =< -y, x, z >$. Find the work required to move an object on the helix $\mathbf{r}(t) =< 2\cos t, 2\sin t, \frac{t}{2\pi} >$ for $0 \leq t \leq 2\pi$.
   Can we use the fundamental theorem or line integrals? why or why not?

5. Use the Divergence Theorem to compute the net outward flux of the vector field $\mathbf{F} =< x, -2y, 3z >$ across the surface of the sphere $x^2 + y^2 + z^2 = 6$.

6. The Amazing Math Kitties, Eva and Archer are trying to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} =< y, xz, -y >$ where C is the ellipse $x^2 + \dfrac{y^2}{4} = 1$ with counterclockwise orientation in the plane $z = 1$. Archer says that he will parameterize the ellipse using $x = \cos t$, $y = 2\sin t$, $z = 1$, Eva says she can use Stoke's Theorem. Which kitty is correct? Are they both correct? If so, calculate the integral using both methods.

B.12 Calculus III HW 12: Due Fri 12/8 Name: ____________

This is your last HW for the year!

1. Please complete the Lewis University evaluation for my Calc III class. In the past it has been posted on Bb, but last semester students received an email from noreply@tk20.com or caleighaoconnell@lewisu.edu with a link to do the evaluations. Once you have completed the evaluation, you will get an email. To get credit and in order for me to calculate whether 90% of the class has completed the evaluation, please forward this email to me. Please give thoughtful, constructive feedback which will help me improve the course. For example, saying "You suck" or "You are great" doesn't provide much feedback for me. Saying "You suck because..." or "You are great because..." Also, remember for everything you like about the course, there is at least one other person who dislikes it, so please let me know what you would like to be kept the same about the course.

   Check one:

   ☐ I completed the evaluation.
   ☐ I have not completed the evaluation.

2. Create a Meme about this course. It can be something about the topic we covered this semester, but it should relate in some way to this course. On the last day of school, we will share all of the memes and vote for the best one. [Note: You can use a meme that already exists if it relates to this course, but it is more fun to create your own.] I will create an assignment in blackboard in which you can upload your Meme or you can print it and attach to this paper. I post my favorite memes on the wall of my office.

3. Create a "good" Archer answer relating to something from what we learned in class this semester. So this should be an incorrect answer (but not a *trivial* incorrect answer) that demonstrates a subtle misconception about a concept or topic in this course. Then write what Eva should say in order to help correct his mistake and explain what the misconception is. Feel free to use the back of this paper.